Hamilton's Ricci Flow

Volume 1

Preliminary Version

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Contents

A note to the reader of the preliminary version vii

Preface ix

Acknowledgments xiii

A guide to Volume 1 xv

Notation xxv

Chapter 1. Basic Riemannian geometry 1
1. Introduction 1
2. Basic conventions and formulas in Riemannian geometry 2
3. Laplacian and Hessian comparison theorems 29
4. Geodesic polar coordinates 37
5. First and second variation of arc length and energy formulas 50
6. Geometric applications of second variation 58
7. Green's function 61
8. Comparison theory for the heat kernel 62
9. Parametrix for the heat equation 64
10. Eigenvalues and eigenfunctions of the Laplacian 69
11. The determinant of the Laplacian 78
12. Monotonicity for harmonic functions and maps 85
13. Lie groups and left invariant metrics 87
14. Bieberbach Theorem 89
15. Compendium of inequalities 94
16. Notes and commentary 95

Chapter 2. Elementary aspects of the Ricci flow equation 101
1. Some geometric flows predating Ricci flow 101
2. Ricci flow and geometrization: a short preview 101
3. Ricci flow and the evolution of scalar curvature 103
4. The maximum principle for heat-type equations 105
5. The maximum principle on noncompact manifolds 107
6. The Einstein-Hilbert functional 111
7. Evolution of geometric quantities - local coordinate calculations 112
8. DeTurck's trick and short time existence 119
9. Notes and commentary 122

Chapter 3. Closed 3-manifolds with positive Ricci curvature 127
1. The maximum principle for tensors 127
2. Hamilton's 1982 theorem 129
3. Evolution of curvature 130
4. The maximum principle for systems 134
5. Gradient of scalar curvature estimate 139
6. Curvature tends to constant 143
7. Exponential convergence to constant curvature of the normalized flow 144
8. Notes and commentary 148

Chapter 4. Ricci solitons and other special solutions 151
1. Types of long existing solutions 151
2. Gradient Ricci solitons 151
3. Gaussian soliton 154
4. Cylinder shrinking soliton 154
5. Cigar steady soliton 155
6. Rosenau solution 158
7. An expanding soliton 160
8. Bryant soliton 162
9. Geometry at spatial infinity of ancient solutions 164
10. Homogeneous solutions 169
11. Isometry group 174
12. Notes and commentary 174

Chapter 5. Analytic results and techniques 177
1. Derivative estimates and long time existence 177
2. Cheeger-Gromov type compactness theorem for Ricci flow 181
3. The Hamilton-Ivey curvature estimate 187
4. Strong maximum principles and splitting theorems 192
5. 3-manifolds with nonnegative curvature 196
6. Manifolds with nonnegative curvature operator 197
7. Notes and commentary 204

Chapter 6. Some miscellaneous techniques for the Ricci, Yamabe and cross curvature flows 207
1. Kazdan-Warner type identities and Yamabe and Ricci solitons 207
2. Andrews' Poincaré type inequality 210
3. The gradient of Hamilton's entropy is the matrix Harnack 212
4. The Yamabe flow and Aleksandrov reflection 213
5. Isoperimetric estimate 220
6. Cross curvature flow 226
7. Notes and commentary 228

Chapter 7. Introduction to singularities 229
1. Dilating about a singularity and taking limits 229

2. Singularity types 231
3. Degenerate neck pinch 237
4. Classification of ancient solutions on surfaces 240
5. Extending noncompact ancient surface solutions to eternal solutions 244
6. Dimension reduction 245
7. Hamilton's partial classification of 3-dimensional singularities 252
8. Some conjectures about ancient solutions 255
9. Notes and commentary 258

Chapter 8. Harnack type estimates 259
1. Li-Yau estimate for the heat equation 259
2. Surfaces with positive curvature 264
3. Harnack estimate on complete surfaces with positive curvature 266
4. Linear trace and interpolated estimates for the Ricci flow on surfaces 268
5. Hamilton's matrix estimate 271
6. Sketch of the proof of the matrix Harnack estimate 275
7. Linear trace Harnack estimate in all dimensions 278
8. A pinching estimate for solutions of the linearized Ricci flow equation 282
9. Notes and commentary 283

Chapter 9. Space-time geometry 287
1. A space-time solution to the Ricci flow for degenerate metrics 287
2. Space-time curvature is the matrix Harnack quadratic 294
3. Potentially infinite metrics and potentially infinite dimensions 295
4. Renormalizing the space-time metric yields the ℓ-length 308
5. Space-time DeTurck's trick and fixing the measure 309
6. Notes and commentary 311

Index 319

Bibliography 325

A note to the reader of the preliminary version

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Preface

The subject of Hamilton's Ricci flow lies in the more general field of geometric flows, which in turn lies in the even more general field of geometric analysis. Ricci flow deforms Riemannian metrics on manifolds by their Ricci tensor, an equation which turns out to exhibit many similarities with the heat equation. Other geometric flows, such as the mean curvature flow of submanifolds demonstrate similar smoothing properties. The aim for many geometric flows is to produce canonical geometric structures by deforming rather general initial data to these structures. Depending on the initial data, the solutions to geometric flows may encounter singularities where at some time the solution can no longer be defined smoothly. For various reasons, in Ricci flow the study of the qualitative aspects of solutions, especially ones which form singularities, is at present more amenable in dimension 3. This is precisely the dimension in which the Poincaré Conjecture was originally stated; the higher dimensional generalizations have been solved by Smale in dimensions at least 5 and by Freedman in dimension 4. Remaining in dimension 3, a vast generalization of this conjecture was proposed by Thurston, called the Geometrization Conjecture, which roughly speaking, says that each closed 3-manifold admits a geometric decomposition, i.e., can be decomposed into pieces which admit complete locally homogeneous metrics.

Hamilton's program is to use Ricci flow to approach this conjecture. Perelman's work aims at completing this program. Through their works one hopes/expects that the Ricci flow may be used to *infer* the existence of a geometric decomposition by taking any initial Riemannian metric on any closed 3-manifold and proving enough analytic, geometric and topological results about the corresponding solution of the Ricci flow with surgery. Note on the other hand that, in this regard, one does *not* expect to need to prove the convergence of the solution to the Ricci flow with surgery to a (possibly disconnected) homogeneous Riemannian manifold. The reason for this is Cheeger and Gromov's theory of collapsing manifolds and its extension to case where the curvature is only bounded from below. Furthermore, if one is only interested in approaching the Poincaré Conjecture, then one does not expect to need the theory of collapsing manifolds. For these geometric and topological reasons, the study of the Ricci flow as an approach to the Poincaré and Geometrization Conjectures are reduced to proving certain analytic and geometric results. In many respects the Ricci flow appears to be a very natural equation and we feel that the study of its analytic and geometric properties is of interest in its own right. Independent of the resolution of the above conjectures, there remain a number of interesting open problems concerning the Ricci flow in dimension 3. In higher dimensions, the situation is perhaps even more interesting in that, in general, much less is known.

So in this book we emphasize the more analytic and geometric aspects of Ricci flow rather than the topological aspects. We also attempt to convey some of the relations and formal similarities between Ricci flow and other geometric flows such as mean curvature flow. The interaction of techniques and ideas between Ricci flow and other geometric flows is a two-way street. So we hope the reader with a more general interest in geometric flows will benefit from the usefulness of ideas originating in Ricci flow to the study of other geometric flows. We have not aimed at completeness, even in the realm of the limited material that we cover. A more extensive coverage of the subject of Ricci flow is planned in the book by Dan Knopf and one of the authors [**153**] and its multi-authored sequel [**143**].

At places we follow the informal style of lecture notes and have attempted to cover some of the basic material in a relatively direct and efficient way. At the same time we take the opportunity to expose the reader to techniques, some of which lie outside of the subject of Ricci flow per se, which they may find useful in pursuing research in Ricci flow. So metaphorically speaking this book is hybrid between rushing to work in the morning on a cold and blistery winter day and a casual stroll through the park on a warm and sunny midsummer afternoon. As much as possible, we have attempted to construct the book so that the chapters, and in some cases, individual sections, are relatively independent. In this way we hope that the book may be used as both an introduction and as a reference. We have endeavored to include some open problems which are aimed at conveying to the reader what are the limits of our current knowledge and to point to some interesting directions. We have also attempted to give the appropriate references so that the reader may further pursue the statements and proofs of the various results. To use real estate jargon, we hope that the references are reliable but they are not guaranteed; in particular, sometimes the references given may not be the first place a particular result is proven.

The year 1982 marked the beginning of Ricci flow with the appearance of Hamilton's paper on 3-manifolds with positive Ricci curvature. Since then, the development of Hamilton's program is primarily scattered throughout a series of several of his papers (see [**79**] for a collection of some Ricci flow papers edited by Cao, Chu, Yau and one of the authors). In Hamilton's papers (sometimes implicitly and sometimes by analogy) a well-developed theory of Ricci flow is created as an approach toward the Geometrization Conjecture. We encourage the reader to go back to these original papers which contain a wealth of information and ideas. Hamilton's program especially takes shape in the three papers [**267**], [**270**] and [**271**]. The first two papers discuss (among other important topics) singularity formation, the classification of singularities, applications of estimates and singularity analysis to the Ricci flow with surgery. The third paper discusses applications of the compactness theorem, minimal surface theory and Mostow rigidity to obtain geometric decompositions of 3-manifolds via Ricci flow under certain assumptions.

The recent spectacular developments due to Grisha Perelman aimed at completing Hamilton's program appear in [**417**] and [**418**]. Again the reader is encouraged to go directly to these sources which contain a plethora of ideas. Perelman's work centers on the further development of singularity and surgery theory. Of primary importance in this regard is Perelman's reduced distance function which has its precursors in the work of Li-Yau on gradient estimates for the heat equation and Hamilton on matrix differential Harnack inequalities for the Ricci flow. A main theme in Perelman's work is the use of comparison geometry, including the understanding of space-time distance, geodesics, and volume to obtain estimates. These estimates build upon in an ingenious way the earlier gradient estimates of Li-Yau-Hamilton and volume comparison theorem of Bishop-Gromov. In a sense, Perelman has further strengthened the bridge between the partial differential equation and comparison approaches to differential geometry in the setting of Ricci flow. For 'finite extinction time', which is aimed at proving the Poincaré Conjecture using neither the theory of collapse nor most of Hamilton's 'non-singular' techniques, see [**419**] and Colding-Minicozzi [**167**]. The relevant work on collapsing manifolds with only lower curvature bounds appear in Perelman [**416**] and Shioya and Yamaguchi [**465**]. For expositions of Perelman's work, see Kleiner-Lott [**318**], Sesum-Tian-Wang [**457**] and Morgan [**386**] (there is also a discussion of [**418**] in Ding [**182**]). We also encourage the reader to consult these excellent expository sources which clarify and fill-in the details for much of Perelman's work.

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This book originated as notes by the authors for lectures on Hamilton's Ricci flow given in China during the summer of 2004. These lecture notes were written during the authors' stays at Fudan, East China Normal, Beijing, and Zhejiang Universities. We soon decided to expanded these notes to cover many more topics including some of Perelman's recent results.

The first author gave lectures at the Beijing University Summer School and the Summer School at the Zhejiang University Center of Mathematical Sciences, and one lecture each at East China Normal University, Fudan University, and the Nankai Institute. The second author gave lectures at the Summer School at the Zhejiang University Center of Mathematical Sciences. The third author gave lectures at Fudan University and a lecture at East China Normal University.

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A guide to Volume 1

Chapter 1. We recall some basic results and facts from Riemannian geometry. The results in this chapter, for the most part, either are used or are analogous to results in the later chapters on Ricci flow and other geometric flows. We begin (§2) with a quick review of metrics, connections, curvature, covariant differentiation and the Lie derivative. Of particular note are the Bianchi identities, which have applications to the formulas we shall derive for solutions of the Ricci flow. We discuss Cartan's method of moving frames since it is a useful technique for computing curvatures, especially in the presence of symmetry. We discuss the decomposition of the curvature tensor into its irreducible components. Since integration by parts is a useful technique in geometric analysis and in particular Ricci flow, we recall the divergence theorem and its consequences. We then recall (§3) the Laplacian and Hessian comparison theorems, which have important analogues in Ricci flow. These are essentially equivalent to the Bishop volume and Rauch comparison theorems. Along these lines, we discuss the Cheeger-Gromoll Splitting Theorem, which relies on the Busemann functions associated to a line being subharmonic (and hence harmonic), and the Mean Value Inequality. Next (§4) we present geodesic polar (spherical) coordinates using the exponential map. This is a convenient way of studying (Hessian) comparison theorems for Jacobi fields, and by taking the determinant, volume (Laplacian) comparison theorems. One may think of these calculations as associated to hypersurfaces (the distance spheres from a point) evolving in their normal directions with unit speed. (More generally, one may consider arbitrary speeds, including the mean curvature flow.) Observe that the Laplacian of the distance function is the radial derivative of the logarithm of the Jacobian, which is the mean curvature of the distance spheres. Similarly, the Hessian of the distance function is the radial derivative of the logarithm of the inner product of Jacobi fields, which is the second fundamental form of the distance spheres. Metric geometry has important implications in Ricci flow. So we discuss (§5) the first and second variation of arc length and energy of paths. An application is Synge's theorem. We include a variational proof of the fact that Jacobi fields minimize the index form. In §6 we discuss the Toponogov comparison theorem and the application of the second variation formula to long geodesics. Next (§7) we discuss the Green's function, which is related to the heat equation and the geometry of the manifold. In §8 we recall some comparison theorems and explicit formulas for the heat kernel on Riemannian manifolds and in particular on constant curvature spaces. Since the Ricci flow evolves metrics and is related to the heat equation, we discuss (§9) the asymptotics of the heat kernel associated to heat operator with respect to an evolving metric. The method is a slight modification of the fixed metric case. We also (§10) recall some basic facts about eigenvalues and eigenfunctions of the Laplacian. In the Ricci flow it useful to study the lowest eigenvalue of certain elliptic (laplace-like) operators. We recall Reilly's formula and its application to estimating the first eigenvalue on manifolds with boundary. In §11 we discuss the definition of the determinant of the Laplacian via the zeta function regularization and compute the difference of determinants on a Riemann surface. The determinant of the Laplacian is an energy functional for the Ricci flow on surfaces. For comparison with later monotonicity formulas, in §12 we recall some monotonicity formulas for harmonic functions and maps. Left-invariant metrics on Lie groups provide nice examples of solutions on Ricci flow which can often be analyzed so we introduce some background material in §13. In §14 we recall the Bieberbach Theorem on the classification of flat manifolds. In the notes and commentary (§16) we review some basic facts about the first and second fundamental forms of hypersurfaces in euclidean space, since we shall later discuss curvature flows of hypersurfaces to compare with Ricci flow.

Chapter 2. We begin the study of Ricci flow proper. We start (§2) with a brief layman description of how Ricci flow approaches the Thurston Geometrization Conjecture. The Ricci flow is like a heat equation for metrics. One quick way to seeing this (§3) is to compute the evolution equation for the scalar curvature using a variation formula we derive later. We get a heat equation with a nonnegative term. Because of this we can apply to maximum principle (§4) to show that the minimum of the scalar curvature increases. The maximum principle extends to complete noncompact manifolds (§5). The variation formula for the scalar curvature yields a short derivation of Einstein's equations as the Euler-Lagrange equation for the total scalar curvature (§6). Next (§7) we carry out the actual computations of the variation of the connection and curvatures. When the variation is minus twice the Ricci tensor, we obtain the evolution equations for the connection, scalar and Ricci curvatures under the Ricci flow. This is the first place we encounter the Lichnerowicz Laplacian which arises in the variation formula for the Ricci tensor. Since the Ricci flow is a weakly parabolic equation, to prove short time existence, we use DeTurck's trick, which shows that it is equivalent to a strictly parabolic equation (§8).

Chapter 3. We describe some of the ingredients in the proof of Hamilton's classification of closed 3-manifolds with positive Ricci curvature using Ricci flow. The maximum principle for tensors (§1) enables us to estimate the Ricci and sectional curvatures. We first show that positive Ricci curvature and Ricci pinching are preserved. Hamilton's theorem (§2) says that under the normalized (volume preserving) Ricci flow on a closed 3-manifold with positive Ricci curvature, the metric converges exponentially fast in every C^k norm to a constant positive sectional curvature metric. Having derived the evolution equations for the Ricci and scalar curvatures, we derive the evolution equation for the full Riemann curvature tensor (§3). This takes the form of a heat equation with a quadratic term on the right hand side. In dimension 3, the form of the quadratic term is especially simple. To control the curvatures it is convenient to generalize the maximum principle for symmetric 2-tensors to a maximum principle for the curvature operator, and more generally, to systems of parabolic equations on a vector bundle (§4). Using this formalism, we show that the pinching of the curvatures improves and tends to constant curvature at points and times where the curvature tends to infinity. This is central estimate in the study of 3-manifolds with positive Ricci curvature. In proving various versions the maximum principle for systems, it is useful to consider the time derivative of the supremum function (§4.1). Once we have a pointwise estimate for the curvatures, we need a gradient estimate for the curvature in order to compare curvatures at different points at the same time (§5). Based on the fact that the pinching estimate breaks the scale-invariance, we obtain a gradient of the scalar curvature estimate which shows that a scale-invariant measure tends to zero at points where the curvature tends to infinity. Combining the estimates of the previous sections, we show that indeed the curvatures tends to constant (§6). With control of the curvatures and assuming derivative estimates derived in a later chapter, we can show that the normalized Ricci flow converges exponentially fast in each C^k norm to a constant positive sectional curvature metric (§7). In the notes and commentary (§8) we state some of the basic evolution equations under the mean curvature flow. These formulas are somewhat analogous to the equations for the Ricci flow.

Chapter 4. We discuss Ricci solitons, homogeneous solutions and other special solutions. From the study of singularity formation we are interested in solutions which exist for all negative time (§1). Some of these solutions exist also for all positive time, which makes them even more special. Of fundamental importance are gradient Ricci solitons (§2). We carefully formulate the basic equations of a gradient Ricci soliton and show that gradient solitons can be put in a canonical form. It is interesting that euclidean space, is not only a steady soliton but also a shrinking and an expanding soliton (§3). For the Ricci flow on low-dimensional manifolds, it is particularly important to consider complete Ricci solitons on surfaces with positive curvature. The cigar soliton (§5) is such a soliton on the euclidean plane, rotationally symmetric, asymptotic to a cylinder at infinity and with curvature decaying exponentially fast. We exhibit the cigar soliton in various coordinate systems. An interesting rotationally symmetric ancient solution on the 2-sphere is the Rosenau solution (§6). As time tends to negative infinity, the Rosenau solution looks like a pair of cigars, one at each end. Indeed, backward (in time) limits at the endpoints yield the cigar soliton. Next we describe an explicit rotationally symmetric expanding soliton on the plane with positive curvature (§7). The curvature also decays exponentially fast as the distance to the origin tends to infinity. Moving up one dimension, we obtain the Bryant soliton (§8). This is a rotationally symmetric steady gradient Ricci soliton on euclidean 3-space with positive sectional curvature. Here the curvature decays inverse linearly and the metric is essentially asymptotic to a parabola at infinity. This means that at infinity the metric dimension reduces to a cylinder in the same sense a parabola in the plane dimension reduces to two parallel lines after dilating about points which tend to infinity. Since complete noncompact ancient solutions are important in the study of singularities, we state some results on the geometry at infinity of such solutions under various hypotheses (§9). Some useful geometric invariants of the geometry at infinity are the asymptotic scalar curvature ratio and the asymptotic volume ratio. Some of the most interesting explicit examples are homogeneous solutions (§10). We consider SU (2) and Nil and analyze the ODE which arises from Ricci flow. Here, since the isometry group acts transitively on the manifold, the Ricci flow reduces to a system of ordinary differential equations. In dimension 3, these systems are for the large part well-understood. Under the Ricci flow, isometries persist (§11). That is, the isometry group is nondecreasing. It is interesting to ask if the isometry group remains constant under the Ricci flow. Analogous to the Ricci flow on surfaces is the curvature shortening flow (CSF) of plane curves (§12). The analogue of the cigar soliton is the grim reaper for the CSF. The Rosenau solution also has an analogue for the CSF.

Chapter 5. We collect some useful analytic results and techniques. Estimates for all of the derivatives of the curvature (§1) in terms of bounds on the curvature and time enable one to show that the solution exists as long as the curvature remains bounds (the long time existence theorem). Thus if a solution forms a singularity in finite time (i.e., cannot be continued past a finite time), then the supremum of the curvature is infinity. The basic tool to study singular solutions is a Cheeger-Gromov type compactness theorem for a sequence of solutions of Ricci flow (§2). We present both the global and local versions of this result. What is needed for this sequence is a curvature bound and an injectivity radius estimate (to prevent collapsing). The sequences we usually consider arise from dilating a singular solution about a sequence of points and times with the times approaching the singularity time. In §2.1 we give a shorter proof of the classification of closed 3-manifolds with positive Ricci curvature using the compactness theorem (and other results such as Perelman's no local collapsing theorem.) Here we only prove sequential convergence instead of exponential convergence. When studying the formation of singularities on a 3-manifold, the Hamilton-Ivey estimate is particularly useful (§3). Roughly speaking, it says that at large curvature points, the largest sectional curvature is positive and much larger than any negative sectional curvature in magnitude. It implies that limits of dilations (which we call singularity models) have nonnegative sectional curvature. Since nonnegative sectional curvature metrics are rather limited geometrically and (especially) topologically, this is crucial first step in the surgery theory for singular 3-dimensional solutions. The Hamilton-Ivey estimate also tells us that ancient 2 and 3-dimensional solutions with bounded curvature have nonnegative sectional curvature. Nonnegative sectional curvature in dimension 3 is a special case of nonnegative curvature operator in all dimensions, a curvature condition which is preserved under the Ricci flow. Such solutions satisfy the strong maximum principle (§4), which says that either the solution has (strictly) positive curvature operator or the holonomy reduces and the image and kernel of the curvature operator are constant in time and invariant under parallel translation. This rigidity result is especially powerful in dimension 3 (§5), where it implies that a simply connected nonnegative sectional curvature solution of the Ricci flow either has positive sectional curvature, splits as the product of a surface solution with positive curvature (which is topologically a 2-sphere or the plane) and a line, or is the flat euclidean space. In higher dimensions (§6), a solution with nonnegative curvature operator either has positive curvature operator, a Kähler manifold with positive curvature operator on $(1,1)$-forms, or is a locally symmetric space. Except in dimension 4, where it was solved by Hamilton, it is an open problem whether a closed Riemannian manifold with positive curvature operator converges under the Ricci flow to a metric with constant positive sectional curvature (spherical space form). A potentially useful result in this regard, due to Tachibana, says that an Einstein metric with positive curvature operator has constant sectional curvature and an Einstein metric with nonnegative curvature operator is locally symmetric. In the notes and commentary we note that the derivative estimates may be improved if we assume bounds on some derivatives of the curvature (§7).

Chapter 6. We deal with identities, inequalities and estimates for various flows including the Ricci flow, Yamabe flow, and the cross curvature flow. It is interesting to compare these techniques. We begin with the Kazdan-Warner and Bourguignon-Ezin identities (§1), from which it follows that a Ricci soliton on the 2-sphere has constant curvature. Then we turn our attention to Andrews's Poincaré type inequality (§2), which holds in arbitrary dimensions. In dimension 2, it implies that Hamilton's entropy is monotone. Another fact (§3) related to Hamilton's entropy is that in the space of metrics on a surface with positive curvature, its gradient is the matrix Harnack quantity, which in dimension 2 is a symmetric 2-tensor. One of the proofs of the convergence of the Ricci on the 2-sphere relies on Ye's gradient estimate for the Yamabe flow of locally conformally flat manifolds with positive Yamabe invariant (§4). This estimate uses the Aleksandrov reflection method to obtain a gradient estimate. In §4.1 we state Leon Simon's asymptotic uniqueness theorem. Another proof, due to Hamilton of the convergence of the Ricci flow on a 2-sphere uses a monotonicity formula for an isoperimetric ratio (§5). We also discuss Hamilton's isoperimetric estimate for Type I singular solutions in dimension three. An interesting fact which follows from the contracted second Bianchi identity, is that the identity map from a Riemannian manifold to itself, where the image manifold has the Ricci tensor as the metric (assuming it is positive), is harmonic (§6). In dimension 3 there is a symmetric curvature tensor, called the cross curvature tensor, which is dual to the Ricci tensor in the following sense. The identity map from a Riemannian 3-manifold to itself, where the domain manifold has the cross curvature tensor as the metric (assuming the sectional curvature is either everywhere negative or everywhere positive), is harmonic. We show two monotonicity formulas for the cross curvature flow which suggest that a negative sectional curvature metric on a closed 3-manifold should converge to a constant negative sectional curvature (hyperbolic) metric. It is interesting that in higher dimensions, there are arbitrarily pinched negative sectional curvature metrics which do not support hyperbolic metrics (§7).

Chapter 7. We begin our study of singularities. To study singularities (§1) one takes dilations about sequences of points and times where the time tends to the singularity time. It is useful to distinguish singularities according to the rate of blow up of the curvature (§2). Type I singularities blow up in finite time at the rate of the standard shrinking sphere. Type II singularities form slower in the sense that in terms of the curvature scale, the time to blow up is longer than that of Type I. On the other hand, as a function of time to blow up the curvature of a Type II singularity is larger than that of a Type I singularity. Given a singularity type, we describe ways of picking sequences of points and times about which to dilate. The limit solutions are ancient solutions. Assuming bounded nonnegative Ricci curvature, we give a lower bound (gap estimate) for the supremum of the curvature as a function of time for ancient solutions. A prototype for a Type II singularity is the degenerate neck pinch (§3). An interesting open problem is to show that Type II singularities indeed exist, a result which is known for the mean curvature flow. Considering ancient solutions on 2-dimensional surfaces (§4), we show using the Harnack inequality that ancient solutions with bounded curvature whose maximum is attained in space in time must be isometric to the cigar soliton. On the other hand, using Hamilton's entropy monotonicity, one can show that Type I ancient solutions are compact and in fact isometric to the round shrinking sphere. Complimentarily, Type II ancient solutions (such as the Rosenau solution) must have a backward limit which is the cigar soliton. Our discussion includes a corollary the classification of ancient κ-solutions on surfaces. An interesting result (§5) is that noncompact ancient solutions on surfaces with time-dependent bounds on the curvature can be extended to eternal solutions. We conjecture that eternal solutions (without assuming the supremum of the curvature is attained) are cigar solitons. To study the geometry at infinity a useful technique is dimension reduction (§6). Here we assume the asymptotic scalar curvature ratio (ASCR) is infinite, that is, the $\limsup$ of the scalar curvature times the square distance to an origin is equal to infinity. Roughly speaking, this says that the scalar curvature has slower than quadratic decay. Using a point picking type argument, we find good sequences of points tending to spatial infinity. Using a result which says that complete manifolds with nonnegative sectional curvature cannot have too many disjoint curvature bumps far from an origin, one can show that when ASCR is infinite, in dimension 3 there exists a limit which splits as the product of a surface solution with a line (§7). By Perelman's no local collapsing theorem, the surface solution cannot be the cigar soliton. By a previous result of Hamilton, the surface must then be a round shrinking 2-sphere. In the case of a Type I ancient solution, we either have a shrinking spherical space form or there exists a backward limit which is a cylinder (2-sphere product with a line). For κ-solutions the latter cannot exist. We conjecture that the κ-noncollapsed condition can be removed. Next we pose some conjectures about ancient solutions in dimensions 2 and 3 (§8). Optimistically, one can hope that the only complete ancient surface solutions with bounded curvature are rotationally symmetric and in fact either the cigar, constant curvature, or the Rosenau solution. In the notes and commentary we note that numerical studies of a degenerate neck pinch have been carried out by Garfinkle and Isenberg (§9).

Chapter 8. We discuss various differential Harnack estimates. These are sharp pointwise derivative estimates which usually enable one compare a solution at different points in space and time. In Ricci flow they have applications toward the classification of singularity models. We begin with the heat equation (§1) and present the seminal Li-Yau estimate for positive solutions. For manifolds with nonnegative Ricci curvature, the estimate is sharp in the sense that equality is obtained for the fundamental solution on euclidean space. Since the Li-Yau estimate has a precursor in the work of Yau on harmonic functions, we discuss the Liouville theorem for complete manifolds with nonnegative Ricci curvature, which relies on a gradient estimate. The Li-Yau estimate has a substantial extension to solutions of the Ricci flow, due to Hamilton. To describe this we begin with surfaces, since the form of the inequality and its proof are much simpler in this case (§2). Ricci solitons again motivate the specific quantities we consider. Analogous to assuming the solution is positive, here we assume the curvature of the surface is positive (§3). A perturbation of these arguments enables one to prove a Harnack estimate for surfaces with variable signed curvature. Interestingly, the above (trace) inequality may be generalized to a matrix inequality which roughly speaking, gives a lower bound for the Hessian of the logarithm of the curvature. In this sense, the Harnack inequalities are somewhat analogous the Laplacian and Hessian comparison theorems of Riemannian geometry discussed in Chapter 1. Next we consider a one-parameter family of Harnack inequalities for the Ricci flow coupled to a linear type heat equation (§4). In one instance we have the Li-Yau inequality, and in another instance we have the linear trace Harnack estimate, which generalizes Hamilton's trace estimate. An open problem is to generalize this to higher dimensions. In the Kähler case, this has be accomplished by Ni. With these preliminaries we move on to Hamilton's celebrated matrix Harnack estimate for solutions of the Ricci flow with nonnegative curvature operator (§5). The specific Harnack under consideration is motivated directly from the consideration of expanding gradient Ricci solitons. The trace inequality (obtained from the matrix inequality by summing over an orthonormal basis) is particularly simple to state and in all dimensions is surprisingly similar to the 2-dimensional inequality. We show that the matrix inequality in dimension 2 is the same as the previous estimate which was in the form of a symmetric 2-tensor being nonnegative. The proof of the matrix estimate depends on a calculation, which at first glance looks quite complicated (§6). However, using the formalism of considering tensors as vector-valued functions on the principal frame bundle (either $\mathrm{GL}(n,\mathbb{R})$ or $\mathrm{O}(n)$) we can simplify the computations. The matrix Harnack quadratic is a bilinear form on the bundle of 1-forms direct sum the bundle of 2-forms. It satisfies a heat-type equation with a quadratic nonlinear term analogous to the heat-type equation satisfied by the Riemann curvature tensor. Taking bases of 1-forms and 2-forms, one can exhibit the quadratic nonlinearity as a sum of squares when the curvature operator is nonnegative. This is the main reason why the matrix Harnack inequality may be proved by a maximum principle argument. The trace Harnack has a generalization to a Harnack inequality for nonnegative definite symmetric 2-tensors satisfying the Lichnerowicz Laplacian heat equation coupled to the Ricci flow (§7). This Harnack generalizes the trace Harnack. Using this we give a simplified proof of Hamilton's result that ancient solutions with nonnegative bounded curvature operator which attain the supremum of their scalar curvatures are steady gradient Ricci solitons. Further investigating the linearized Ricci flow (§8), we give a pinching estimate for solutions to the linearized Ricci flow on closed 3-manifolds. An open problem is to find applications of this general estimate, perhaps in conjunction with the linear trace Harnack or other new estimates. In the notes and commentary (§9) we briefly discuss the matrix Harnack for the heat equation, the Harnack for the mean curvature flow, and some tools for calculating evolution equations associated to Ricci flow.

Chapter 9. We discuss various space-time geometries which are rather similar culminating in Perelman's metric on space-time product with large dimensional and large radius spheres. We begin with Hamilton's notion of the Ricci flow for degenerate metrics and the space-time connection of Chu and one of the authors (§1). This connection is compatible with the degenerate space-time metric and as a pair they satisfy the Ricci flow for degenerate metrics. We state the formulas for the Riemann and Ricci tensors of the space-time connection and observe curvature identities which suggest that the metric-connection pair is a Ricci soliton. We observe (§2) that the space-time Riemann curvature tensor is the matrix Harnack quadratic and the space-time Ricci tensor is the trace Harnack quadratic. Next we take the product of space-time with Einstein metric solutions of the Ricci flow (§3). We also introduce a scalar parameter into the definition of the potentially infinite space-time metric. We calculate the Levi-Civita connections of these metrics and observe that they essentially tend to the space-time connection defined in §1. It is particularly interesting that the space-time Laplacians tend to the heat (forward or backward) operator (depending on the sign the scalar parameter.) This fact depends on the dimension tending to infinity. Next we compute the Riemann and Ricci curvature tensors of the potentially infinite metrics. We observe that when the scalar parameter is -1, the metric tends to Ricci flat (this is due to Perelman). This is related to the developments on the ℓ-function discussed in a later chapter. We also recall the observation, again due to Perelman, that the metrics are essentially potentially Ricci soliton (at least he observed this when the parameter is either -1 or 1.) Renormalizing the space-time length functional, we obtain the ℓ-function (§4). Not only is the space-time metric and connection related to the matrix Harnack, it is also related to the linear trace Harnack (§5). Here we need to make some modifications to describe the relation. An auxiliary function f is introduced and its definition is related to Perelman's idea of fixing the measure which he introduced when defining his energy and entropy. In the notes and commentary we discuss a space-time formulation of flows of hypersurfaces (§6) and its relation with Andrews' Harnack inequalities.

Notation

Here we list some of the notation which we use throughout the book.

(1) Area area of a surface or volume of a hypersurface
(2) dA area form (volume form in dimension 2)
(3) ASCR asymptotic scalar curvature ratio
(4) AVR asymptotic volume ratio
(5) $B(p, r)$ ball of radius r centered at p
(6) bounded curvature = bounded *sectional* curvature
(7) const constant
(8) $\operatorname{Cut}(p)$ cut locus of p
(9) Γ_{ij}^{k} Christoffel symbols
(10) ∇ covariant derivative
(11) $\doteqdot$ defined to be equal to
(12) d distance
(13) div divergence
(14) $\cdot$ either dot product or multiplication, depending on the context
(15) int interior
(16) Δ, Δ_L, Δ_d Laplacian, Lichnerowicz and Hodge Laplacians
(17) L length
(18) H mean curvature
(19) $\operatorname{Hess}(f)$ or $\nabla\nabla f$ Hessian of f
(20) $g(X, Y) = \langle X, Y \rangle$ metric or inner product
(21) log natural logarithm
(22) $\mathcal{F}$, $\mathcal{W}$ Perelman's energy, entropy functional
(23) $\mathcal{I}$ a time interval for the Ricci flow
(24) $\mathcal{J}$ a time interval for the backward Ricci flow
(25) ℓ reduced distance or ℓ-function
(26) $\mathcal{L}$ Lie derivative or $\mathcal{L}$-length, depending on the context
(27) ν unit outward normal
(28) ODE / ODI ordinary differential equation / inequality
(29) PDE partial differential equation
(30) RHS right-hand side
(31) R, Rc, Rm scalar, Ricci and Riemann curvature tensors
(32) II or h second fundamental form
(33) $S(p, r)$ distance sphere
(34) tr or Trace trace
(35) Vol volume of a manifold

(36) $d\mu$ volume form
(37) $d\sigma$ volume form on the boundary or on a hypersurface
(38) ω_n volume of the unit euclidean n-ball
(39) $n\omega_n$ volume of the unit euclidean $(n-1)$-sphere
(40) $W^{1,2}$ Sobolev space of first derivatives in L^2
(41) $X^\flat$ dual 1-form to the vector field X
(42) $\alpha^\natural$ dual vector field to the 1-form α

CHAPTER 1

Basic Riemannian geometry

Wake up, Neo... The Matrix has you... Follow the white rabbit.

In this first chapter we present some basic Riemannian geometry with an emphasis on the materials which are related to the techniques encountered in the study of the Ricci flow. The work of Hamilton has emphasized the Bochner formulas, maximum principles, and pointwise monotonicity formulas. The recent work of Perelman brings into this picture of the Ricci flow: space-time and comparison geometry, and integral and pointwise monotonicity formulas. The reader familiar with basic Riemannian geometry may skip the following sections and proceed directly to Chapter 2.

Sections 2-6 discuss the basic definitions of metric, connection, curvature, covariant differentiation, geodesics, Jacobi fields, exponential map, Hessian, Laplacian, volume, Jacobian, and their properties. We emphasize local coordinate calculations and review Bochner formulas, integration by parts, and comparison theory such as the Laplacian and Hessian comparison theorems, the volume and Rauch comparison theorems and Toponogov comparison theorem. Sections 7-12 review some aspects of the laplace and heat equations including their fundamental solutions (Green's function and heat kernel), spectral theory, and harmonic functions and maps. The rest of the chapter considers some miscellaneous topics such as Lie groups, the Bieberbach theorem and a compendium of some of the basic inequalities that we use later in the book.

1. Introduction

In Riemannian geometry one often asks the following question. Given a restriction on the curvature of a Riemannian manifold, what topological conditions follow? **Myers' theorem** is a good example of this. It says that a complete n-dimensional manifold with Ricci curvature[1] bounded below by a positive constant $(n-1)\,k$ has diameter at most $\pi/\sqrt{k}$. Topological consequences of this are that the manifold is compact and has finite fundamental group. Another nice example is the **Cartan-Hadamard theorem** which

[1]For the definition of Ricci curvature and other basic concepts in Riemannian geometry, see section 2 below.

says that a simply connected, complete Riemannian manifold with nonpositive sectional curvature is diffeomorphic to $\mathbb{R}^n$ and each exponential map is a diffeomorphism.

In the subject of Ricci flow one starts with a Riemannian manifold and deforms the metric in the direction of minus the Ricci tensor. The underlying differentiable manifold stays the same. Here one hopes to prove that the geometry of the metric improves as it evolves. The first success, in fact the main theorem of the seminal paper [**255**] by Richard Hamilton in 1982, has the following topological consequence.

THEOREM 1.1 (Hamilton 1982 - 3-manifolds with positive Ricci curvature). *If (M^3, g) is a closed 3-manifold with positive Ricci curvature, then it is diffeomorphic to a **spherical space form**. That is, M^3 admits a metric with constant positive sectional curvature.*

Here and throughout this book, closed means compact without boundary. More ambitiously, one would like to use Ricci flow to prove the following **Geometrization Conjecture** .

CONJECTURE 1.2 (Thurston Geometrization). *Every closed 3-manifold admits a geometric decomposition.*

A corollary of the Geometrization Conjecture is the **Poincaré Conjecture**, which says that every simply connected closed topological 3-manifold is homeomorphic to the 3-sphere and which is one of the Millennium prize problems. We refer to the survey papers of Thurston [**487**] and Scott [**450**] for background on the Geometrization Conjecture. For more recent surveys of the geometrization conjecture, see the notes and commentary at the end of this chapter.

2. Basic conventions and formulas in Riemannian geometry

Let (M^n, g) be a **Riemannian manifold**. That is, M^n is an n-dimensional differentiable manifold and g is a **Riemannian metric**. A Riemannian metric g (also denoted by $\langle \ , \ \rangle$) is a smoothly varying inner product on the tangent spaces (Riemann 1854.) Equivalently, we may think of g either as a positive-definite section of the bundle of symmetric (covariant) 2-tensors $T^*M \otimes_S T^*M$ or as positive-definite bilinear maps $g(x) : T_xM \times T_xM \to \mathbb{R}$, for all $x \in M$. Here, $T^*M \otimes_S T^*M$ is the subspace of $T^*M \otimes T^*M$ generated by elements of the form $X \otimes Y + Y \otimes X$. The metric g defines an infinitesimal notion of length and angle. The length of a tangent vector X is defined by:

$$|X| \doteqdot g(X, X)^{1/2}$$

and the angle between two nonzero tangent vectors X and Y is defined by:

$$\angle(X, Y) \doteqdot \cos^{-1}\left(\frac{\langle X, Y \rangle}{|X|\,|Y|}\right).$$

Let $\left\{x^i\right\}$ be local coordinates in a neighborhood U of some point of M. In U the vector fields $\left\{\partial/\partial x^i\right\}$ form a local basis for TM and the 1-forms $\left\{dx^i\right\}$ form a dual basis for T^*M, that is, $dx^i(\partial/\partial x^j) = \delta^i_j$. The metric may be then written in local coordinates as

$$g = g_{ij} dx^i \otimes dx^j,$$

where $g_{ij} \doteqdot g(\partial/\partial x^i, \partial/\partial x^j)$ and we have used, as we will throughout the book, the Einstein summation convention. We shall often denote the metric g (and tensors in general) by its components g_{ij}.

Given a smooth immersion $\varphi : N^m \to M^n$ and a metric g on M^n we can pull back g to a metric on N

$$(\varphi^* g)(V, W) \doteqdot g(\varphi_* V, \varphi_* W).$$

Note that if $\{y^\alpha\}$ and $\left\{x^i\right\}$ are local coordinates on N and M respectively, then

$$(\varphi^* g)_{\alpha\beta} = g_{ij} \frac{\partial \varphi^i}{\partial y^\alpha} \frac{\partial \varphi^j}{\partial y^\beta}$$

where $(\varphi^* g)_{\alpha\beta} \doteqdot (\varphi^* g)\left(\partial/\partial y^\alpha, \partial/\partial y^\beta\right)$ and $\varphi^i \doteqdot x^i \circ \varphi$. More generally, given any covariant p-tensor α on M^n and a smooth map $\varphi : N^m \to M^n$, we define the pull back of α to N by

$$(\varphi^* \alpha)(X_1, \ldots, X_p) = \alpha(\varphi_* X_1, \ldots, \varphi_* X_p)$$

for all $X_1, \ldots, X_p \in T_y N$. If φ is a diffeomorphism, then the pull back of contravariant tensors is defined as the push forward by φ^{-1}.

Recall the **Levi-Civita connection** (or **covariant derivative**) $\nabla : TM \times C^\infty(TM) \to C^\infty(TM)$ is the unique connection on TM that is compatible with the metric and torsion free:

$$X(g(Y, Z)) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z) \tag{1.1}$$

$$\nabla_X Y - \nabla_Y X = [X, Y], \tag{1.2}$$

where it is customary to write $\nabla_X Y$ for $(\nabla Y)(X) = \nabla(X, Y)$ (note ∇Y is a $(1,1)$-tensor) and

$$[X, Y] f \doteqdot X(Yf) - Y(Xf)$$

defines the **Lie bracket** acting on functions. From this one can show by taking a linear combination of the above equations with permutations of the vector fields X, Y and Z that

$$\begin{aligned} 2g(\nabla_X Y, Z) &= X(g(Y, Z)) + Y(g(X, Z)) - Z(g(X, Y)) \\ &\quad + g([X, Y], Z) - g([X, Z], Y) - g([Y, Z], X). \end{aligned} \tag{1.3}$$

The differentiable structure on M^n enables us to define the directional derivatives of functions which, besides the linearity properties, satisfy the product rule $X(fh) = hX(f) + fX(h)$. Covariant differentiation, which is defined by using the Riemannian metric, tells us how to differentiate vector fields. Equation (1.1) is the product rule (compatibility with the metric) and

(1.2) is a compatibility condition with the differentiable structure (torsion-free).

EXERCISE 1.3. *Let ∇^g denote the Levi-Civita connection of the metric g. Show that for any constant $c > 0$ and metric g, $\nabla^{cg} = \nabla^g$.*

Let $\{x^i\}_{i=1}^n$ be a local coordinate system defined in an open set U in M^n.[2] The **Christoffel symbols** are defined in U by $\nabla_{\partial/\partial x^i}\partial/\partial x^j \doteqdot \Gamma_{ij}^k \partial/\partial x^k$. Here and throughout the book, we follow the **Einstein summation convention** of summing over repeated indices. By (1.3) and $\left[\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right] = 0$ we see that they are given by

$$\Gamma_{ij}^k = \frac{1}{2} g^{k\ell} \left(\frac{\partial}{\partial x^i} g_{j\ell} + \frac{\partial}{\partial x^j} g_{i\ell} - \frac{\partial}{\partial x^\ell} g_{ij} \right). \tag{1.4}$$

EXERCISE 1.4. *Let $\{x^i\}$ and $\{y^\alpha\}$ be coordinate functions on a common open set. Using $g_{\alpha\beta} = g_{ij} \frac{\partial x^i}{\partial y^\alpha} \frac{\partial x^j}{\partial y^\beta}$ show that*

$$\Gamma_{\alpha\beta}^\gamma \frac{\partial x^k}{\partial y^\gamma} = \Gamma_{ij}^k \frac{\partial x^i}{\partial y^\alpha} \frac{\partial x^j}{\partial y^\beta} + \frac{\partial^2 x^k}{\partial y^\alpha \partial y^\beta}.$$

SOLUTION. (See also §I.7 of [**198**].) Let $\Phi_\alpha^i = \frac{\partial x^i}{\partial y^\alpha}$ so that $g_{\alpha\beta} = g_{ij} \Phi_\alpha^i \Phi_\beta^j$ and $g^{\gamma\delta} = g^{k\ell} \left(\Phi^{-1}\right)_k^\gamma \left(\Phi^{-1}\right)_\ell^\delta$. We compute

$$\begin{aligned}
\Gamma_{\alpha\beta}^\gamma \Phi_\gamma^k &= \frac{1}{2} g^{k\ell} \left(\Phi^{-1}\right)_\ell^\delta \left(\frac{\partial}{\partial y^\alpha} \left(g_{jm} \Phi_\beta^j \Phi_\delta^m \right) + \frac{\partial}{\partial y^\beta} \left(g_{im} \Phi_\alpha^i \Phi_\delta^m \right) - \frac{\partial}{\partial y^\delta} \left(g_{ij} \Phi_\alpha^i \Phi_\beta^j \right) \right) \\
&= \Phi_\alpha^i \Phi_\beta^j \Gamma_{ij}^k + \frac{1}{2} g^{k\ell} \left(g_{j\ell} \frac{\partial}{\partial y^\alpha} \Phi_\beta^j + g_{i\ell} \frac{\partial}{\partial y^\beta} \Phi_\alpha^i - g_{ij} \left(\left(\Phi^{-1}\right)_\ell^\delta \frac{\partial}{\partial y^\delta} \Phi_\alpha^i \right) \Phi_\beta^j \right) \\
&\quad + \frac{1}{2} g^{k\ell} \left(\Phi^{-1}\right)_\ell^\delta \left(g_{jm} \Phi_\beta^j \frac{\partial}{\partial y^\alpha} \Phi_\delta^m + g_{im} \Phi_\alpha^i \frac{\partial}{\partial y^\beta} \Phi_\delta^m - g_{ij} \Phi_\alpha^i \frac{\partial}{\partial y^\delta} \Phi_\beta^j \right) \\
&= \Phi_\alpha^i \Phi_\beta^j \Gamma_{ij}^k + \frac{\partial^2 x^k}{\partial y^\alpha \partial y^\beta}.
\end{aligned}$$

where we used

$$\frac{\partial}{\partial y^\alpha} \Phi_\beta^k = \frac{\partial}{\partial y^\beta} \Phi_\alpha^k = \frac{\partial^2 x^k}{\partial y^\alpha \partial y^\beta}, \quad \frac{\partial}{\partial y^\alpha} \Phi_\delta^i = \frac{\partial}{\partial y^\delta} \Phi_\alpha^i, \quad \frac{\partial}{\partial y^\beta} \Phi_\delta^j = \frac{\partial}{\partial y^\delta} \Phi_\beta^j.$$

EXERCISE 1.5. *Verify that if (M^n, g) is a Riemannian manifold, $\varphi : N^m \to M^n$ is an immersion, and $\{y^\alpha\}$ and $\{x^i\}$ are local coordinates on N and M respectively, then*

$$\Gamma\left(\varphi^* g\right)_{\alpha\beta}^\gamma \frac{\partial \varphi^k}{\partial y^\gamma} = \left(\Gamma_{ij}^k \circ \varphi \right) \frac{\partial \varphi^i}{\partial y^\alpha} \frac{\partial \varphi^j}{\partial y^\beta} + \frac{\partial^2 \varphi^k}{\partial y^\alpha \partial y^\beta}$$

where $\varphi^i \doteqdot x^i \circ \varphi$.

[2] The reason we often carry out computations in local coordinates, as compared to moving frames, is that under the Ricci flow the metric is time-dependent. However, in some cases using an evolving moving frame simplifies computations.

The covariant derivative defines the notion of parallel translation along a path. In particular a vector field X along a path $\gamma : (a, b) \to M^n$ is **parallel** if

$$\nabla_{\dot\gamma} X = 0$$

along γ. We say that a path γ is a **geodesic** if the unit tangent vector field is parallel along γ:

$$\nabla_{\dot\gamma}\left(\frac{\dot\gamma}{|\dot\gamma|}\right) = 0.$$

A geodesic has constant speed if $|\dot\gamma|$ is constant along γ; in this case $\nabla_{\dot\gamma}\dot\gamma = 0$. As we shall see in section 5, the shortest path between two points is a geodesic and locally geodesics minimize length.

EXERCISE 1.6. *Show that if X is parallel along a path γ, then $|X|^2$ is constant along γ.*

SOLUTION.

$$\nabla_{\dot\gamma} |X|^2 = 2\left\langle \nabla_{\dot\gamma} X, X\right\rangle = 0.$$

The **Riemann curvature** $(3, 1)$**-tensor** Rm is defined by

$$\operatorname{Rm}(X, Y) Z \doteqdot \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z. \tag{1.5}$$

One easily checks that for any function f

$$\operatorname{Rm}(fX, Y) Z = \operatorname{Rm}(X, fY) Z = \operatorname{Rm}(X, Y)(fZ) = f \operatorname{Rm}(X, Y) Z. \tag{1.6}$$

Thus Rm is indeed a tensor. It is also nice to define

$$\nabla^2_{X,Y} Z \doteqdot \nabla_X \nabla_Y Z - \nabla_{\nabla_X Y} Z$$

so that

$$\operatorname{Rm}(X, Y) Z = \nabla^2_{X,Y} Z - \nabla^2_{Y,X} Z, \tag{1.7}$$

$$\nabla^2_{fX,Y} Z = \nabla^2_{X,fY} Z = f \nabla^2_{X,Y} Z, \tag{1.8}$$

and

$$\nabla^2_{X,Y}(fZ) = f \nabla^2_{X,Y} Z + Y(f)\ \nabla_X Z + X(f)\ \nabla_Y Z - ((\nabla_X Y) f)\ Z + X(Y(f))\ Z \tag{1.9}$$

for any function f. Note that from (1.8), (1.9) and (1.7) we can immediately derive (1.6).

REMARK 1.7. *The bracket measures the noncommutativity of the directional derivative acting on functions, whereas* Rm *measures the noncommutativity of covariant differentiation acting on vector fields.*

The **components** of the $(3, 1)$-tensor Rm are defined by

$$\operatorname{Rm}\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right)\frac{\partial}{\partial x^k} \doteqdot R_{ijk}^{\ell}\frac{\partial}{\partial x^\ell}$$

and $R_{ijk\ell} \doteqdot g_{\ell m} R_{ijk}^m$; note

$$R_{ijk\ell} = \operatorname{Rm}\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}, \frac{\partial}{\partial x^k}, \frac{\partial}{\partial x^\ell}\right) \doteqdot \left\langle \operatorname{Rm}\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right) \frac{\partial}{\partial x^k}, \frac{\partial}{\partial x^\ell} \right\rangle$$

are the components of Rm as a $(4,0)$-tensor. Some basic symmetries of the Riemann curvature tensor are

$$R_{ijk\ell} = -R_{jik\ell} = -R_{ij\ell k} = R_{k\ell ij}. \tag{1.10}$$

If $P \subset T_x M^n$ is a 2-plane, then the **sectional curvature** of P is defined by

$$K(P) \doteqdot \langle \operatorname{Rm}(e_1, e_2) e_2, e_1 \rangle$$

where $\{e_1, e_2\}$ is an orthonormal basis of P ; this definition is independent of the choice of such a basis.

EXERCISE 1.8. *Show that if X and Y are any two vectors spanning P, then*

$$K(P) = \frac{\langle \operatorname{Rm}(X,Y) Y, X \rangle}{|X|^2 |Y|^2 - \langle X, Y \rangle^2}.$$

EXERCISE 1.9. *Using (1.5) and (1.4) show that*

$$\boxed{R_{ijk}^\ell = \partial_i \Gamma_{jk}^\ell - \partial_j \Gamma_{ik}^\ell + \Gamma_{jk}^p \Gamma_{ip}^\ell - \Gamma_{ik}^p \Gamma_{jp}^\ell \,.} \tag{1.11}$$

The **Ricci tensor** Rc is the trace of the Riemann curvature tensor:

$$\operatorname{Rc}(Y, Z) \doteqdot \operatorname{trace}(X \mapsto \operatorname{Rm}(X,Y) Z).$$

In terms of an orthonormal frame $\{e_a\}_{a=1}^n$, i.e., a frame with $g(e_a, e_b) = \delta_{ab}$, we have $\operatorname{Rc}(Y,Z) = \sum_{a=1}^n \operatorname{Rm}(e_a, Y) Z \cdot e_a$. Its components, defined by $R_{ij} \doteqdot \operatorname{Rc}\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right)$, are given by

$$R_{jk} = \sum_{i=1}^n R_{ijk}^i.$$

The **Ricci curvature** of a line $L \subset T_x M^n$ is defined by

$$\operatorname{Rc}(L) \doteqdot \operatorname{Rc}(e_1, e_1)$$

where $e_1 \in T_x M^n$ is a unit vector spanning L.

The **scalar curvature** is the trace of the Ricci tensor: $R = \sum_{a=1}^n \operatorname{Rc}(e_a, e_a)$. In local coordinates, $R = g^{ij} R_{ij}$; here $g^{ij} \doteqdot \left(g^{-1}\right)_{ij}$ is the inverse matrix. Note that we can globally define a **metric g^{-1} on the cotangent bundle** by $g^{-1} \doteqdot g^{ij} \frac{\partial}{\partial x^i} \otimes \frac{\partial}{\partial x^j}$ in any local coordinate system.

EXERCISE 1.10. *Show that g^{-1} is well-defined.*

EXERCISE 1.11 (Scaling properties of the curvatures). *Given a metric g and a positive constant C, show that* $\operatorname{Rm}_{(3,1)}(Cg) = \operatorname{Rm}_{(3,1)}(g)$ *(as a* $(3,1)$*-tensor),* $\operatorname{Rm}_{(4,0)}(Cg) = C \operatorname{Rm}_{(4,0)}(g)$, $\operatorname{Rc}(Cg) = \operatorname{Rc}(g)$, *and* $R(Cg) = C^{-1} R(g)$.
Hint: use Exercise 1.3.

EXERCISE 1.12 (Geometric interpretation of tracing). *Show that the trace of a symmetric* 2*-tensor* α *is given by the following formula:*

$$\operatorname{Trace}_g(\alpha) = \frac{1}{\omega_n} \int_{S^{n-1}} \alpha(V, V)\, d\sigma(V)$$

where S^{n-1} *is the unit* $(n-1)$*-sphere,* $n\omega_n$ *its volume, and* $d\sigma$ *its volume form. From this show for any unit vector* U, *that* $\frac{1}{n-1}\operatorname{Rc}(U,U)$ *is the average of the sectional curvatures of planes containing the vector* U. *Similarly,* $\frac{1}{n}R(p)$ *is the average of* $\operatorname{Rc}(U,U)$ *over all unit vectors* $U \in S^{n-1} \subset T_pM^n$.

SOLUTION. There exists an orthonormal basis $\{e_i\}_{i=1}^n$ such that $\alpha = \sum_{i=1}^n \lambda_i e_i^* \otimes e_i^*$. Furthermore, $\operatorname{Trace}_g(\alpha) = \sum_{i=1}^n \lambda_i$ and

$$\frac{1}{\omega_n} \int_{S^{n-1}} \langle V, e_i \rangle^2 \, d\sigma(V) = 1.$$

See also [**218**].

REMARK 1.13. *As discussed in the introduction, a basic problem is to understand the effect curvature conditions (such as the curvature having a given sign) have on the topology and geometry of a manifold. The sectional curvature tells us the most, the scalar curvature the least, and the Ricci curvature is somewhere in between.*

Just as important as differentiating functions and vector fields is differentiating tensors. By requiring that the covariant differentiation commutes with contractions and that the product (Leibnitz) rule holds, one defines covariant differentiation acting on tensors. In particular, acting on $(0,s)$-tensors, we define covariant differentiation by

$$\nabla_X : C^\infty(\otimes^s TM) \to C^\infty(\otimes^s TM),$$

where

$$\nabla_X(Z_1 \otimes \cdots \otimes Z_s) \doteqdot \sum_{i=1}^s Z_1 \otimes \cdots \otimes \nabla_X Z_i \otimes \cdots \otimes Z_s.$$

The covariant derivative of an (r,s)-tensor α is then defined by:
(1.12)

$$\nabla_X \alpha(Y_1, \ldots, Y_r) \doteqdot \nabla_X \left(\alpha(Y_1, \ldots, Y_r)\right) - \sum_{i=1}^r \alpha\left(Y_1, \ldots, \nabla_X Y_i, \ldots, Y_r\right),$$

where each term is an element of $C^\infty(\otimes^s TM)$. Let $\otimes^{r,s} M = (\otimes^r TM^*) \otimes (\otimes^s TM)$. The covariant derivative may be considered as:

$$\nabla : C^\infty(\otimes^{r,s} M) \to C^\infty(\otimes^{r+1,s} M),$$

where

$$\nabla\alpha(X, Z_1, \ldots, Z_r) \doteqdot \nabla_X \alpha(Z_1, \ldots, Z_r),$$

or equivalently,

$$\nabla\alpha = \sum_{i=1}^n \nabla_i \alpha \otimes dx^i.$$

In this way we may square the covariant derivative operator:

$$\nabla^2 : C^\infty(\otimes^{r,s} M) \to C^\infty(\otimes^{r+2,s} M),$$

which is given by

$$\begin{aligned}\nabla^2 \alpha\,(X, Y, Z_1, \ldots, Z_r) &= \nabla_X\,(\nabla\alpha)\,(Y, Z_1, \ldots, Z_r) \\ &= [\nabla_X\,(\nabla\alpha(Y)) - \nabla\alpha\,(\nabla_X Y)]\,(Z_1, \ldots, Z_r) \\ &= \nabla_X \nabla_Y \alpha\,(Z_1, \ldots, Z_r) - \nabla_{\nabla_X Y}\alpha\,(Z_1, \ldots, Z_r)\,.\end{aligned}$$

Using the notation

$$\nabla^2_{X,Y}\alpha\,(Z_1, \ldots, Z_r) \doteqdot \nabla^2\alpha\,(X, Y, Z_1, \ldots, Z_r)\,,$$

we may rewrite this as

$$\nabla^2_{X,Y}\alpha = \nabla_X \nabla_Y \alpha - \nabla_{\nabla_X Y}\alpha.$$

We now have (1.1) is equivalent to

$$\nabla g = 0 \tag{1.13}$$

(i.e., the metric is parallel).[3] As a matter of notation, we let $\nabla_X \beta$ denote the covariant derivative of a tensor β with respect to a vector X. We shall usually use local coordinates to express the components of tensors and follow the classical index notation (Ricci calculus). If β is an (r, s)-tensor, then we define the components $\nabla_i \beta^{k_1 \cdots k_s}_{j_1 \cdots j_r}$ of the covariant derivative $\nabla\beta$ of β by

$$\nabla_i \beta^{k_1 \cdots k_s}_{j_1 \cdots j_r} \frac{\partial}{\partial x^{k_1}} \otimes \cdots \otimes \frac{\partial}{\partial x^{k_s}} \doteqdot \left(\nabla_{\partial/\partial x^i}\beta\right) \left(\frac{\partial}{\partial x^{j_1}}, \ldots, \frac{\partial}{\partial x^{j_r}}\right).$$

We then have

$$\begin{aligned}\nabla_i \beta^{k_1 \cdots k_s}_{j_1 \cdots j_r} &= \frac{\partial}{\partial x^i} \beta^{k_1 \cdots k_s}_{j_1 \cdots j_r} - \sum_{m=1}^{r} \sum_{\ell=1}^{n} \Gamma^\ell_{i j_m} \beta^{k_1 \cdots k_s}_{j_1 \cdots j_{m-1} \ell j_{m+1} \cdots j_r} \\ &\quad + \sum_{p=1}^{s} \sum_{q=1}^{n} \Gamma^{k_p}_{iq} \beta^{k_1 \cdots k_{p-1} q k_{p+1} \cdots k_s}_{j_1 \cdots j_r}.\end{aligned} \tag{1.14}$$

For example

$$\begin{aligned}\nabla_i R_{jk} &\doteqdot (\nabla \operatorname{Rc}) \left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}, \frac{\partial}{\partial x^k}\right) \\ &= \left(\nabla_{\frac{\partial}{\partial x^i}} \operatorname{Rc}\right) \left(\frac{\partial}{\partial x^j}, \frac{\partial}{\partial x^k}\right) = \frac{\partial}{\partial x^i} R_{jk} - \Gamma^\ell_{ij} R_{\ell k} - \Gamma^\ell_{ik} R_{j\ell}\end{aligned}$$

and

$$\begin{aligned}\nabla_i R_{jk\ell m} &\doteqdot \left(\nabla_{\partial/\partial x^i} \operatorname{Rm}\right) \left(\frac{\partial}{\partial x^j}, \frac{\partial}{\partial x^k}, \frac{\partial}{\partial x^\ell}, \frac{\partial}{\partial x^m}\right) \\ &= \frac{\partial}{\partial x^i} R_{jk\ell m} - \Gamma^p_{ij} R_{pk\ell m} - \Gamma^p_{ik} R_{jp\ell m} - \Gamma^p_{i\ell} R_{jkpm} - \Gamma^p_{im} R_{jk\ell p}.\end{aligned}$$

[3] In general we say that a tensor is parallel if its covariant derivative is zero.

EXERCISE 1.14. *Show that*

$$\nabla_i \nabla_j f \doteqdot (\nabla\nabla f)\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right) = \frac{\partial^2 f}{\partial x^i \partial x^j} - \Gamma_{ij}^k \frac{\partial f}{\partial x^k}$$

and more generally for a 1-form X

$$\nabla_i X_j = \frac{\partial}{\partial x^i} X_j - \Gamma_{ij}^k X_k.$$

Besides (1.10) there are additional symmetries the Riemann curvature tensor satisfies. The **first** and **second Bianchi identities** are:

$$R_{ijk\ell} + R_{jki\ell} + R_{kij\ell} = 0 \tag{1.15}$$

$$\nabla_i R_{jk\ell m} + \nabla_j R_{ki\ell m} + \nabla_k R_{ij\ell m} = 0. \tag{1.16}$$

The **(twice) contracted second Bianchi identity** is

$$\boxed{2g^{ij} \nabla_i R_{jk} = \nabla_k R.} \tag{1.17}$$

This is equivalent to the **Einstein tensor** $\mathrm{Rc} - \frac{1}{2} R g$ being divergence-free:

$$\operatorname{div}\left(\mathrm{Rc} - \frac{1}{2} R g\right) = 0$$

(see (1.58) below for the definition of divergence). Formula (1.17) follows from multiplying the second Bianchi identity (1.16) by $g^{im} g^{j\ell}$.

EXERCISE 1.15 (Once contracted 2nd Bianchi identity). *Show that by multiplying (1.16) by* g^{im} *and summing (that is, contracting once), we have*

$$g^{im} \nabla_i R_{jk\ell m} = \nabla_j R_{k\ell} - \nabla_k R_{j\ell}. \tag{1.18}$$

That is, the divergence of Rm *is the exterior covariant derivative of* Rc *considered as a* 1*-form with values in the tangent bundle.*

As we shall see in the next section, the Bianchi identities are related to the diffeomorphism invariance of the curvature.

A Riemannian manifold (M^n, g) has **constant sectional curvature** if the sectional curvature of every 2-plane is the same. That is, there exists $k \in \mathbb{R}$ such that for every $x \in M$ and 2-plane $P \subset T_x M$, $K(P) = k$. Similarly we say that a metric has **constant Ricci curvature** if the Ricci curvature of every line is the same.

EXERCISE 1.16 (Schur).

(1) *Using (1.17), show that if* g *is an Einstein metric:* $R_{ij} = \frac{1}{n} R g_{ij}$ *and* $n \geq 3$, *then* R *is a constant. Note that the condition* $R_{ij} = \frac{1}{n} R g_{ij}$ *says that the Ricci curvatures depend only on the point and not on the line at the point. The result of this exercise says that in this case, if* $n \geq 3$, *then the Ricci curvatures also do not depend on the point.*

(2) *Using the second Bianchi identity (1.16), show that if* $n \geq 3$ *and the sectional curvatures at each point are independent of the 2-plane, that is, if*

$$R_{ijk\ell} = \frac{R}{n(n-1)}\left(g_{i\ell}g_{jk} - g_{ik}g_{j\ell}\right),$$

then R *is a constant.*

2.1. Lie derivative. Let α be a tensor and X a complete vector field generating a global 1-parameter group of diffeomorphisms φ_t (the following definition extends to the case where X is not complete and only defines local 1-parameter groups of diffeomorphisms). The **Lie derivative** of α with respect to X is defined by

$$\mathcal{L}_X \alpha \doteqdot \lim_{t\to 0} \frac{1}{t}\left(\alpha - (\varphi_t)_* \alpha\right). \tag{1.19}$$

Here $(\varphi_t)_* : T_pM^n \to T_{\varphi_t(p)}M^n$ is the differential of φ_t. It acts on the cotangent bundle by $(\varphi_t)_* = \left(\varphi_t^{-1}\right)^* : T_p^*M^n \to T_{\varphi_t(p)}^*M^n$. We can then naturally extend the action of $(\varphi_t)_*$ to the tensor bundles of M^n, which is used in (1.19).

REMARK 1.17. *If* $\psi : M^n \to N^m$ *is a map and* α *is an* $(r,0)$ *tensor on* N^m*, then*

$$(\psi^*\alpha)(Y_1,\ldots,Y_r) \doteqdot \alpha(\psi_* Y_1,\ldots,\psi_* Y_r).$$

The Lie derivative, which measures the infinitesimal lack of diffeomorphism invariance of a tensor with respect to a 1-parameter group of diffeomorphisms generated by a vector field, has the following properties:

(1) If f is a function, then $\mathcal{L}_X f = Xf$.
(2) If Y is a vector field, then $\mathcal{L}_X Y = [X,Y]$.
(3) If α and β are tensors, then $\mathcal{L}_X(\alpha\otimes\beta) = (\mathcal{L}_X\alpha)\otimes\beta + \alpha\otimes(\mathcal{L}_X\beta)$.
(4) If α is an $(r,0)$-tensor, then for any vector fields $X, Y_1,\ldots,Y_r$

$$\begin{aligned}(\mathcal{L}_X\alpha)(Y_1,\ldots,Y_r) &= X(\alpha(Y_1,\ldots,Y_r)) \\ &\quad - \sum_{i=1}^n \alpha(Y_1,\ldots,Y_{i-1},[X,Y_i],Y_{i+1},\ldots,Y_r) \\ &= (\nabla_X\alpha)(Y_1,\ldots,Y_r) \\ &\quad + \sum_{i=1}^n \alpha(Y_1,\ldots,Y_{i-1},\nabla_{Y_i}X,Y_{i+1},\ldots,Y_r).\end{aligned} \tag{1.20}$$

For example, if α is a 2-tensor, then

$$(\mathcal{L}_X\alpha)_{ij} = \nabla_X\alpha_{ij} + g^{k\ell}\left(\nabla_i X_k \alpha_{\ell j} + \nabla_j X_k \alpha_{i\ell}\right).$$

EXERCISE 1.18. *Given a diffeomorphism* $\varphi : M^n \to M^n$*, we have* $\varphi^* : T_{\varphi(p)}^*M^n \to T_p^*M^n$*. The pull back acts on the tangent bundle by* $\varphi^* =$

$\left(\varphi^{-1}\right)_* : T_{\varphi(p)}M^n \rightarrow T_pM^n$. *These actions extend to the tensor bundles of* M^n. *Show that definition (1.19) is equivalent to*

$$\mathcal{L}_X\alpha = \lim_{t\to 0} \frac{1}{t}\left(\varphi_t^*\alpha - \alpha\right) = \left.\frac{d}{dt}\right|_{t=0} \varphi_t^*\alpha.$$

Recall that the gradient of a function f with respect to the metric g is defined by $g\left(\operatorname{grad}_g f, X\right) \doteqdot Xf = df\left(X\right)$. In other words, $\operatorname{grad}_g f$ is the metric dual of the 1-form df. Throughout this book we shall also use the notation ∇f to denote both df and $\operatorname{grad}_g f$.

EXERCISE 1.19 (Lie derivative of the metric). *Using (1.20), show that the Lie derivative of the metric is given by*

$$(\mathcal{L}_X g)(Y_1, Y_2) = g\left(\nabla_{Y_1}X, Y_2\right) + g\left(Y_1, \nabla_{Y_2}X\right) \tag{1.21}$$

and that in local coordinates this implies

$$\boxed{(\mathcal{L}_X g)_{ij} = \nabla_i X_j + \nabla_j X_i.}$$

In particular, if f *is a function, then*

$$\boxed{\left(\mathcal{L}_{\operatorname{grad}_g f}\, g\right)_{ij} = 2\nabla_i\nabla_j f.} \tag{1.22}$$

REMARK 1.20. *Formula (1.22) is useful when considering gradient Ricci solitons.*

EXERCISE 1.21. *Show that for any diffeomorphism* $\varphi : M^n \rightarrow M^n$, *tensor* α, *and vector field* X,

$$\varphi^*\left(\mathcal{L}_X\alpha\right) = \mathcal{L}_{\varphi^*X}\left(\varphi^*\alpha\right), \tag{1.23}$$

and if $f : M^n \rightarrow \mathbb{R}$, *then*

$$\varphi^*\left(\operatorname{grad}_g f\right) = \operatorname{grad}_{\varphi^*g}\left(f\circ\varphi\right). \tag{1.24}$$

SOLUTION. Let $\psi\left(t\right)$ denote the 1-parameter group of diffeomorphisms generated by X:

$$\begin{aligned}\varphi^*\left(\mathcal{L}_X\alpha\right) &= \varphi^*\left(\lim_{t\to 0}\frac{\psi\left(t\right)^*\alpha - \alpha}{t}\right)\\ &= \lim_{t\to 0}\frac{\left(\varphi^{-1}\circ\psi\left(t\right)\circ\varphi\right)^*\varphi^*\alpha - \varphi^*\alpha}{t} = \mathcal{L}_Y\left(\varphi^*\alpha\right)\end{aligned}$$

where Y is the vector field generating the 1-parameter group of diffeomorphisms $\varphi^{-1}\circ\psi\left(t\right)\circ\varphi$. Now

$$\begin{aligned}Y\left(x\right) &= \left.\frac{d}{dt}\right|_{t=0}\varphi^{-1}\circ\psi\left(t\right)\circ\varphi\left(x\right) = \left(\varphi^{-1}\right)_*\left.\frac{d}{dt}\right|_{t=0}\psi\left(t\right)\circ\varphi\left(x\right)\\ &= \left(\varphi^{-1}\right)_*\left(X\left(\varphi\left(x\right)\right)\right) = \left(\varphi^*X\right)\left(x\right).\end{aligned}$$

For any $x \in M^n$ and $X \in T_{\varphi(x)}M^n$ we have

$$\left\langle \varphi^* \left(\operatorname{grad}_g f\right), \varphi^* X \right\rangle_{\varphi^* g} (x) = \left\langle \operatorname{grad}_g f, X \right\rangle_g (\varphi (x)) = (Xf)(\varphi(x)) = (\varphi^* X)(f \circ \varphi)(x).$$

REMARK 1.22. *If $\varphi(t) : M^n \to M^n$ is the 1-parameter family of diffeomorphisms and α is a tensor, then*

$$\frac{\partial}{\partial t} \left(\varphi(t)^* \alpha\right) = \mathcal{L}_{X(t)} \varphi(t)^* \alpha \tag{1.25}$$

where

$$X(t_0) \doteqdot \left.\frac{\partial}{\partial t}\right|_{t=t_0} \left(\varphi(t_0)^{-1} \circ \varphi(t)\right) = \left(\varphi(t_0)^{-1}\right)_* \left.\frac{\partial}{\partial t}\right|_{t=t_0} \varphi(t).$$

Here we have not assumed that $\varphi(t)$ is a group.

We say that a diffeomorphism $\psi : (M^n, g) \to (N^n, h)$ is an **isometry** if $\psi^* h = g$. Two metrics are said to be **isometric** if there is an isometry between them. Geometrically such metrics are indistinguishable. We say that a vector field X on (M^n, g) is **Killing** if $\mathcal{L}_X g = 0$. Recall that a vector field X is **complete** if there is 1-parameter group of diffeomorphisms $\{\varphi_t\}_{t \in \mathbb{R}}$ generated by X. If M^n is closed, then any smooth vector field is complete. If X is a complete Killing vector field, then the 1-parameter group of diffeomorphisms φ_t that it generates is a 1-parameter group of isometries of (M^n, g).

EXERCISE 1.23 (2nd Bianchi identity from diffeomorphism invariance of curvature).

(1) *Prove the Jacobi identity*

$$[X,[Y,Z]] + [Y,[Z,X]] + [Z,[X,Y]] = 0$$

for vector fields X, Y, Z as follows. Let $\varphi_t : M^n \to M^n$ be the one-parameter group of diffeomorphisms generated by X and take the time derivative at $t = 0$ of the 'invariance of the Lie bracket under diffeomorphism' equation:

$$\varphi_t^* [Y, Z] = [\varphi_t^* Y, \varphi_t^* Z].$$

(2) (Hilbert, Kazdan [**311**]) *Similarly, prove the (contracted) second Bianchi identities by considering the diffeomorphism invariance of the scalar curvature and Riemannian curvature tensor.*

(a) *More precisely, to obtain the contracted second Bianchi identity (1.17) apply (2.4) to the equation:*

$$DR_g \left(\mathcal{L}_X g\right) = \mathcal{L}_X R = \nabla_i R X^i \tag{1.26}$$

where $DR_g(\mathcal{L}_X g)$ denotes the linearization of R_g in the direction $\mathcal{L}_X g$;

(b) *to prove the second Bianchi identity (1.16) apply (3.13) to:*

$$D \operatorname{Rm}_g \left(\mathcal{L}_X g\right) = \mathcal{L}_X \operatorname{Rm}. \tag{1.27}$$

SOLUTION. 2a) From (2.4) and (1.26) we have

$$\nabla R \cdot X = DR_g(\mathcal{L}_X g) = -2\Delta \operatorname{div}(X) + \operatorname{div}(\operatorname{div} \mathcal{L}_X g) - \langle \mathcal{L}_X g, \operatorname{Rc} \rangle . \tag{1.28}$$

Now

$$\begin{aligned}(\operatorname{div} \mathcal{L}_X g)_i &= \nabla_j (\nabla_j X_i + \nabla_i X_j) \\ &= \Delta X_i + \nabla_i \operatorname{div}(X) + R_{ik} X_k\end{aligned}$$

and

$$\operatorname{div}(\operatorname{div} \mathcal{L}_X g) = \operatorname{div}(\Delta X) + \Delta \operatorname{div}(X) + \operatorname{div}(\operatorname{Rc}(X)) .$$

The first term on the LHS may be rewritten as

$$\begin{aligned}\operatorname{div}(\Delta X) &= \nabla_i \nabla_j \nabla_j X_i \\ &= \nabla_j \nabla_i \nabla_j X_i \\ &= \Delta \operatorname{div}(X) + \operatorname{div}(\operatorname{Rc}(X))\end{aligned}$$

(check the second equality). Hence

$$\operatorname{div}(\operatorname{div} \mathcal{L}_X g) = 2\Delta \operatorname{div}(X) + 2 \operatorname{div}(\operatorname{Rc}(X)) .$$

Substituting this in (1.28) we obtain

$$\begin{aligned}\nabla R \cdot X &= 2 \operatorname{div}(\operatorname{Rc}(X)) - \langle \mathcal{L}_X g, \operatorname{Rc} \rangle \\ &= 2 \operatorname{div}(\operatorname{Rc}) \cdot X.\end{aligned}$$

Since X is arbitrary, we conclude that $\nabla R = 2 \operatorname{div}(\operatorname{Rc})$.

2b) From (3.14) we deduce

$$\begin{aligned}\frac{\partial}{\partial s} R_{ijk\ell} = &\frac{1}{2} (\nabla_i \nabla_k v_{j\ell} - \nabla_i \nabla_\ell v_{jk} - \nabla_j \nabla_k v_{i\ell} + \nabla_j \nabla_\ell v_{ik}) \\ &+ \frac{1}{2} (R_{ijkq} v_{q\ell} + R_{ijq\ell} v_{qk})\end{aligned}$$

(note the slight change in the formula due to the lowering of an index in Rm .) Thus (1.27) implies

$$\begin{aligned}&\frac{1}{2} \begin{pmatrix} \nabla_i \nabla_k (\nabla_j X_\ell + \nabla_\ell X_j) - \nabla_i \nabla_\ell (\nabla_j X_k + \nabla_k X_j) \\ -\nabla_j \nabla_k (\nabla_i X_\ell + \nabla_\ell X_i) + \nabla_j \nabla_\ell (\nabla_i X_k + \nabla_k X_i) \end{pmatrix} \\ &+ \frac{1}{2} (R_{ijkm} (\nabla_m X_\ell + \nabla_\ell X_m) + R_{ijm\ell} (\nabla_m X_k + \nabla_k X_m)) \\ &= D \operatorname{Rm}_g (\mathcal{L}_X g) = \mathcal{L}_X \operatorname{Rm} \\ &= X^m \nabla_m R_{ijk\ell} + R_{mjk\ell} \nabla_i X^m + R_{imk\ell} \nabla_j X^m + R_{ijm\ell} \nabla_k X^m + R_{ijkm} \nabla_\ell X^m\end{aligned} \tag{1.29}$$

First we recognize $\nabla_m R_{ijk\ell}$ as a potential second Bianchi identity term. Next we look at the terms on the first two lines of the above equation. For example the first terms on the top two lines are

$$\begin{aligned}\nabla_i\nabla_k\nabla_j X_\ell - \nabla_j\nabla_k\nabla_i X_\ell &= \nabla_i\nabla_j\nabla_k X_\ell - \nabla_j\nabla_i\nabla_k X_\ell \\ &\quad - \nabla_i\left(R_{kj\ell m}X_m\right) + \nabla_j\left(R_{ki\ell m}X_m\right) \\ &= -R_{ijkm}\nabla_m X_\ell - R_{ij\ell m}\nabla_k X_m \\ &\quad - \nabla_i\left(R_{kj\ell m}X_m\right) + \nabla_j\left(R_{ki\ell m}X_m\right)\end{aligned}$$

Similarly (switch k and ℓ in the above)

$$\begin{aligned}-\nabla_i\nabla_\ell\nabla_j X_k + \nabla_j\nabla_\ell\nabla_i X_k &= R_{ij\ell m}\nabla_m X_k + R_{ijkm}\nabla_\ell X_m \\ &\quad + \nabla_i\left(R_{\ell jkm}X_m\right) - \nabla_j\left(R_{\ell ikm}X_m\right).\end{aligned}$$

Next we look at

$$\nabla_i\nabla_k\nabla_\ell X_j - \nabla_i\nabla_\ell\nabla_k X_j = -\nabla_i\left(R_{k\ell jm}X_m\right)$$

and

$$-\nabla_j\nabla_k\nabla_\ell X_i + \nabla_j\nabla_\ell\nabla_k X_i = \nabla_j\left(R_{k\ell im}X_m\right).$$

Substituting the above into (1.29), we get

$$\begin{aligned}&- R_{ijkm}\nabla_m X_\ell - R_{ij\ell m}\nabla_k X_m - \nabla_i\left(R_{kj\ell m}X_m\right) + \nabla_j\left(R_{ki\ell m}X_m\right) \\ &+ R_{ij\ell m}\nabla_m X_k + R_{ijkm}\nabla_\ell X_m + \nabla_i\left(R_{\ell jkm}X_m\right) - \nabla_j\left(R_{\ell ikm}X_m\right) \\ &- \nabla_i\left(R_{k\ell jm}X_m\right) + \nabla_j\left(R_{k\ell im}X_m\right) \\ &+ R_{ijkm}\left(\nabla_m X_\ell + \nabla_\ell X_m\right) + R_{ijm\ell}\left(\nabla_m X_k + \nabla_k X_m\right) \\ &= 2X^m\nabla_m R_{ijk\ell} + 2R_{mjk\ell}\nabla_i X^m + 2R_{imk\ell}\nabla_j X^m + 2R_{ijm\ell}\nabla_k X^m + 2R_{ijkm}\nabla_\ell X^m.\end{aligned}$$

Simplifying this yields

$$\begin{aligned}0 &= \nabla_i\left[\left(R_{jk\ell m} + R_{\ell jkm} + R_{k\ell jm}\right)X_m\right] + \nabla_j\left[\left(R_{ki\ell m} + R_{i\ell km} + R_{\ell kim}\right)X_m\right] \\ &\quad - 2\left(\nabla_i R_{jmk\ell} + \nabla_m R_{ijk\ell} + \nabla_j R_{mik\ell}\right)X_m.\end{aligned}$$

Now we choose X so that at a point $X = 0$ and $\nabla_i X_j = \delta_{ij}$. Then

$$\begin{aligned}0 &= \left(R_{jk\ell i} + R_{\ell jki} + R_{k\ell ji}\right) + \left(R_{ki\ell j} + R_{i\ell kj} + R_{\ell kij}\right) \\ &= 2\left(R_{jk\ell i} + R_{\ell jki} + R_{k\ell ji}\right)\end{aligned}$$

which implies the first Bianchi identity. Hence the first line vanishes and we conclude

$$\left(\nabla_i R_{jmk\ell} + \nabla_m R_{ijk\ell} + \nabla_j R_{mik\ell}\right)X_m = 0.$$

Since X is arbitrary, we obtain the second Bianchi identity.

Since the commutation of covariant derivatives acting on vector fields defines the Riemann curvature tensor, the commutation of covariant derivatives acting on tensors may be expressed in terms of the curvature. We shall find particularly useful the standard **commutation formulas** (**Ricci identities**):

$$(1.30)\qquad \left(\nabla_i\nabla_j - \nabla_j\nabla_i\right)\alpha_{k_1\cdots k_r} = -\sum_{\ell=1}^{r} R_{ijk_\ell}^m \alpha_{k_1\cdots k_{\ell-1}mk_{\ell+1}\cdots k_r}.$$

In particular, if α is a 1-form, then

$$(\nabla_i\nabla_j - \nabla_j\nabla_i)\,\alpha_k = -R_{ijk}^{\ell}\alpha_\ell.$$

If β is a $(2,0)$-tensor, then:

$$\nabla_i\nabla_j\beta_{k\ell} - \nabla_j\nabla_i\beta_{k\ell} = -R_{ijk}^{p}\,\beta_{p\ell} - R_{ij\ell}^{p}\,\beta_{kp}. \tag{1.31}$$

Throughout most of this book we shall find it convenient to compute in local coordinates rather than in an orthonormal (moving) frame. The reason for this is that under the Ricci flow the metric is evolving and we can choose a fixed (time-independent) coordinate system whereas an orthonormal frame, in order to stay orthonormal with respect to $g(t)$ must evolve. An exception to this is Uhlenbeck's trick, which we discuss right after Lemma 3.11.

EXERCISE 1.24. *Show directly using the Killing vector field equation that the vector space of Killing vector fields is a Lie algebra.*

SOLUTION. Using $\nabla_j X_i = -\nabla_i X_j$ and $\nabla_j Y_i = -\nabla_i Y_j$, we compute

$$\begin{aligned}&\nabla_i\left(X_k\nabla_k Y_j - Y_k\nabla_k X_j\right) + \nabla_j\left(X_k\nabla_k Y_i - Y_k\nabla_k X_i\right)\\ &= X_k\left(\nabla_i\nabla_k Y_j + \nabla_j\nabla_k Y_i\right) - Y_k\left(\nabla_i\nabla_k X_j + \nabla_j\nabla_k X_i\right)\\ &= -X_k Y_\ell\left(R_{ikj\ell} + R_{jki\ell}\right) + Y_k X_\ell\left(R_{ikj\ell} + R_{jki\ell}\right) = 0.\end{aligned}$$

REMARK 1.25. *The Bianchi identities and the commutation formulas for covariant differentiation are often used in the calculations of the equations governing the evolution of geometric quantities under the Ricci flow.*

EXERCISE 1.26.

$$\begin{aligned}\nabla_i\nabla_j\alpha_{k_1\cdots k_r}^{\ell_1\cdots\ell_s} - \nabla_j\nabla_i\alpha_{k_1\cdots k_r}^{\ell_1\cdots\ell_s} &= -\sum_{h=1}^{r}\sum_{p=1}^{n} R_{ijk_h}^{p}\,\alpha_{k_1\cdots k_{h-1}\,p\,k_{h+1}\cdots k_r}^{\ell_1\cdots\ell_s}\\ &\quad+\sum_{h=1}^{s}\sum_{p=1}^{n} R_{ijp}^{\ell_h}\alpha_{k_1\cdots k_r}^{\ell_1\cdots\ell_{h-1}\;p\ell_{h+1}\cdots\ell_s}.\end{aligned} \tag{1.32}$$

2.2. Decomposition of the curvature tensor. The Riemann curvature $(4,0)$-tensor is a section of the bundle $\wedge^2 M^n \otimes_S \wedge^2 M^n$. Moreover, by the first Bianchi identity, Rm is a section of the subbundle $\ker(b)$, the kernel of the linear map:

$$b : \wedge^2 M^n \otimes_S \wedge^2 M^n \to \wedge^3 M^n \otimes_S T^* M^n$$

defined by

$$b(\Omega)(X,Y,Z,W) \doteqdot \frac{1}{3}\left(\Omega(X,Y,Z,W) + \Omega(Y,Z,X,W) + \Omega(Z,X,Y,W)\right).$$

We shall call $\mathrm{C}\,M \doteqdot \ker(b)$ the bundle of curvature tensors. For every $x \in M^n$, the fiber $\mathrm{C}_x\,M$ has the structure of an $\mathrm{O}(T_x^* M)$-module, given by

$$\times : \mathrm{O}(T_x^* M) \times \mathrm{C}_x\,M \to \mathrm{C}_x\,M$$

where

$$A \times (\alpha\otimes\beta\otimes\gamma\otimes\delta) \doteqdot A\alpha\otimes A\beta\otimes A\gamma\otimes A\delta$$

for $A \in \mathrm{O}\left(T_x^* M\right)$ and $\alpha, \beta, \gamma, \delta \in T_x^* M$. As an $\mathrm{O}\left(T_x^* M\right)$ representation space, $\mathrm{C}_x\, M$ has a natural decomposition into its irreducible components. This yields a corresponding decomposition of the Riemann curvature tensor. To describe this, it will be convenient to consider the Kulkarni-Nomizu product

$$\odot : S^2 M \times S^2 M \to \mathrm{C}\, M$$

defined by

$$(\alpha \odot \beta)_{ijk\ell} \doteqdot \alpha_{i\ell}\beta_{jk} + \alpha_{jk}\beta_{i\ell} - \alpha_{ik}\beta_{j\ell} - \alpha_{j\ell}\beta_{ik}.$$

Here $S^2 M = T^* M \otimes_S T^* M$ is the bundle of symmetric 2-tensors. The irreducible decomposition of $\mathrm{C}_x\, M$ as an $\mathrm{O}\left(T_x^* M\right)$-module is given by:

$$\mathrm{C}\, M = \mathbb{R} g \odot g \oplus \left(S_0^2 M \odot g\right) \oplus \mathrm{W}\, M \tag{1.33}$$

where $S_0^2 M$ is the bundle of symmetric, trace-free 2-tensors and

$$\mathrm{W}\, M \doteqdot \ker(b) \cap \ker(c)$$

is the bundle of Weyl curvature tensors. Here

$$c : \wedge^2 M^n \otimes_S \wedge^2 M^n \to S^2 M$$

is the contraction map defined by

$$c(\Omega)(X, Y) \doteqdot \sum_{i=1}^{n} \Omega(e_i, X, Y, e_i).$$

The irreducible decomposition of $\mathrm{C}\, M$ yields the following irreducible decomposition of the Riemann curvature tensor:

$$\mathrm{Rm} = f g \odot g \oplus (h \odot g) \oplus W$$

where $f \in C^\infty(M)$, $h \in C^\infty\left(S_0^2 M\right)$, and $W \in C^\infty(\mathrm{W}\, M)$. Taking the contraction c of this equation implies

$$R_{jk} = 2(n-1) f\ g_{jk} + (n-2) h_{jk}.$$

On the other hand, taking two contractions, we find that

$$R = 2n(n-1) f.$$

Therefore we have

$$\mathrm{Rm} = -\frac{R}{(n-1)(n-2)} g \odot g \frac{1}{n-2} \mathrm{Rc} \odot g + \mathrm{Weyl} \tag{1.34}$$

$$= \frac{R}{2n(n-1)} g \odot g + \frac{1}{n-2} \overset{\circ}{\mathrm{Rc}} \odot g + \mathrm{Weyl} \tag{1.35}$$

where $\overset{\circ}{\mathrm{Rc}} \doteqdot \mathrm{Rc} - \frac{R}{n} g$ is the traceless Ricci tensor and Weyl is the Weyl tensor, which is *defined* by (1.34). The Weyl tensor has the same algebraic symmetries as the Riemann curvature tensor and in addition the Weyl tensor is totally trace-free, all of its traces are zero, including:

$$g^{ik} W_{ijk\ell} = 0$$

and conformally invariant:

$$\operatorname{Weyl}(ug) = u \operatorname{Weyl}(g) \tag{1.36}$$

for any positive smooth function u on M.

If $n \leq 3$, then the Weyl tensor vanishes. In particular, if $n = 2$ we have

$$R_{ijk\ell} = \frac{1}{2} R \left(g_{i\ell} g_{jk} - g_{ik} g_{j\ell} \right), \tag{1.37}$$

and $R_{ij} = \frac{1}{2} R g_{ij}$. When $n = 3$ we have

$$R_{ijk\ell} = R_{i\ell} g_{jk} + R_{jk} g_{i\ell} - R_{ik} g_{j\ell} - R_{j\ell} g_{ik} - \frac{1}{2} R \left(g_{i\ell} g_{jk} - g_{ik} g_{j\ell} \right). \tag{1.38}$$

One way of seeing (1.37) is that the bundle $\wedge^2 M^2 \otimes_S \wedge^2 M^2$ has rank one and both sides of formula (1.37) are sections of this bundle and have the same double trace R. Equation (1.38) follows from (1.34) and the vanishing of the Weyl tensor.

EXERCISE 1.27. *Show that the Weyl tensor vanishes when $n = 3$.*

SOLUTION. There are only two possible types of nonzero components of W. Either there are 3 distinct indices such as W_{1231} or there are two distinct indices such as W_{1221}. First we compute, using the trace-free property,

$$W_{1231} = -W_{2232} - W_{3233} = 0.$$

Next, we have

$$W_{1221} = -W_{2222} - W_{3223} = -W_{3223} = W_{3113} = -W_{2112} = -W_{1221}$$

which implies $W_{1221} = 0$.

EXERCISE 1.28. *Show that if $\tilde{g} = e^{2f} g$ for some function f, then*

$$\tilde{R}_{ijk}^{\ell} = R_{ijk}^{\ell} - a_i^{\ell} g_{jk} - a_{jk} \delta_i^{\ell} + a_{ik} \delta_j^{\ell} + a_j^{\ell} g_{ik}$$

where

$$a_{ij} = \nabla_i \nabla_j f - \nabla_i f \nabla_j f + \frac{1}{2} |\nabla f|^2 g_{ij}.$$

That is, as $(4,0)$-tensors,

$$e^{-2f} \widetilde{\operatorname{Rm}} = \operatorname{Rm} - a \odot g.$$

From this deduce (1.36).

HINT. First show that

$$\tilde{R}_{ijk}^{\ell} = R_{ijk}^{\ell} + \nabla_i A_{jk}^{\ell} - \nabla_j A_{ik}^{\ell} + A_{jk}^{m} A_{im}^{\ell} - A_{ik}^{m} A_{jm}^{\ell}$$

where

$$A_{ij}^{k} = \tilde{\Gamma}_{ij}^{k} - \Gamma_{ij}^{k} = \nabla_i f \ \delta_j^{k} + \nabla_j f \ \delta_i^{k} - \nabla^k f \ g_{ij}.$$

EXERCISE 1.29. *From (1.33) we have the (reducible) decomposition:*

$$CM \cong \left(S^2 M \odot g \right) \oplus \mathrm{W}\, M. \tag{1.39}$$

(1) *Show that*

$$\mathrm{Rm} = \frac{1}{n-2} S \odot g + \mathrm{Weyl}$$

where

$$S \doteqdot \mathrm{Rc} - \frac{R}{2(n-1)} g$$

is the ***Schouten tensor****.*

(2) *Show that if* $n \geq 3$*, then*

$$\nabla^{\ell} W_{ijk\ell} = \frac{n-3}{n-2} B_{ijk}$$

where

$$\begin{aligned} B_{ijk} &\doteqdot \nabla_i S_{jk} - \nabla_j S_{ik} \\ &= \nabla_i R_{jk} - \nabla_j R_{ik} - \frac{1}{2(n-1)} \left(\nabla_i R g_{jk} - \nabla_j R g_{ik} \right) \end{aligned}$$

is the ***Bach tensor****.*

From the above exercise we see that for $n \geq 4$, if the Weyl tensor of (M^n, g) vanishes, then the Bach tensor vanishes. We also see that when $n = 3$, the Weyl tensor always vanishes.

EXERCISE 1.30. *Show that when* $n = 3$*, if* $\tilde{g} = ug$*, then*

$$\tilde{B}_{ijk} = u^{3/2} B_{ijk}.$$

A Riemannian manifold (M^n, g) is said to be **locally conformally flat** if for every point $p \in M^n$, there exists a local coordinate system $\{x^i\}$ in a neighborhood U of p such that

$$g_{ij} = g\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j} \right) = v \cdot \delta_{ij}$$

for some function v defined on U. When $n = 2$, every Riemannian manifold is locally conformally flat.

PROPOSITION 1.31 (Weyl 1918, Schouten 1921). *A Riemannian manifold* (M^n, g) *is locally conformally flat if and only if*

(1) *for* $n \geq 4$ *the Weyl tensor vanishes,*
(2) *for* $n = 3$ *the Bach tensor vanishes.*

PROOF. By the conformal invariance of the Weyl tensor, it is clear that if (M^n, g) is locally conformally flat, then the Weyl tensor vanishes. Conversely, if the Weyl tensor vanishes, then the equation that the metric $\tilde{g} = e^{2f} g$ is flat:

$$\widetilde{\mathrm{Rm}} = 0$$

is equivalent to

$$0 = e^{-2f}\ \widetilde{\mathrm{Rm}} = \mathrm{Rm} - a \odot g$$
$$= \left(\frac{1}{n-2} \left(\mathrm{Rc} - \frac{1}{2(n-1)} R g \right) - a \right) \odot g. \tag{1.40}$$

Since the map

$$\odot : S^2 M \to \mathrm{C}\, M$$

defined by

$$\odot(h) \doteqdot h \odot g$$

is injective, (1.40) is equivalent to

$$\frac{1}{n-2} \left(\mathrm{Rc} - \frac{1}{2(n-1)} R g \right) = a,$$

that is,

$$\nabla_i \nabla_j f = b_{ij} + \nabla_i f \nabla_j f - \frac{1}{2} |\nabla f|^2 g_{ij}, \tag{1.41}$$

where

$$b_{ij} = \frac{1}{n-2} \left(R_{ij} - \frac{1}{2(n-1)} R g_{ij} \right). \tag{1.42}$$

Proposition 1.31 is now a consequence of the following, which gives the condition for when the flat metric equation for $\tilde{g}$ is locally solvable.

LEMMA 1.32. *Equation (1.41) is locally solvable if and only if the following integrability condition is satisfied:*

$$\nabla_k b_{ij} = \nabla_i b_{kj}, \tag{1.43}$$

that is, if and only if

$$\nabla_k R_{ij} - \frac{1}{2(n-1)} \nabla_k R g_{ij} = \nabla_i R_{kj} - \frac{1}{2(n-1)} \nabla_i R g_{kj}.$$

PROOF. To solve (1.41) it is necessary and sufficient to find a 1-form X such that locally

$$\nabla_i X_j = b_{ij} + X_i X_j - \frac{1}{2} |X|^2 g_{ij}. \tag{1.44}$$

This is because by the symmetry of the RHS of (1.44),

$$\nabla_i X_j = \nabla_j X_i,$$

which implies that locally X is the exterior derivative of a some function f. We rewrite (1.44) as

$$\partial_i X_j = c_{ij} \tag{1.45}$$

where

$$c_{ij} = \Gamma_{ij}^k X_k + b_{ij} + X_i X_j - \frac{1}{2} |X|^2 g_{ij}.$$

The integrability condition for (1.45) is $\partial_k c_{ij} = \partial_i c_{kj}$. This can be seen as follows. For each index j consider equation (1.45) as

$$d(X_j) = c_j \tag{1.46}$$

where, for each j, c_j is the 1-form defined by $c_j = \sum_{i=1}^n c_{ij} dx^i$. The necessary and sufficient conditions to solve (1.46) locally are

$$0 = (dc_j)_{ik} = \partial_i c_{kj} - \partial_k c_{ij}.$$

Finally, (1.45) is equivalent to

$$\nabla_k c_{ij} - \nabla_i c_{kj} = -R^p_{kij} X_p$$

which is equivalent to (1.43). □

COROLLARY 1.33. *If (M^n, g) has constant sectional curvature, then (M^n, g) is locally conformally flat.*

The proposition above and the decomposition (1.39) enables one to easily verify certain examples are locally conformally flat. In particular, we have:

COROLLARY 1.34.

(1) *If (N, g^N) and (P, g^P) are Riemannian manifolds such that*

$$\operatorname{sect}\left(g^N\right) \equiv C \quad \textit{and} \quad \operatorname{sect}\left(g^P\right) \equiv -C,$$

where $C \in \mathbb{R}$, then their Riemannian product $(N \times P, g^{N\times P})$ is locally conformally flat.

(2) *If (N, g^N) has $\operatorname{sect}\left(g^N\right) \equiv C$, then the Riemannian product $(N \times \mathbb{R}, g^N + dt^2)$ is locally conformally flat.*

PROOF. 1) The Riemann curvature tensor of the product is:

$$\begin{aligned}\operatorname{Rm}^{N\times P} &= \operatorname{Rm}^N + \operatorname{Rm}^P = \frac{1}{2} g^N \odot g^N - \frac{1}{2} g^P \odot g^P \\ &= \frac{1}{2}\left(g^N - g^P\right) \odot \left(g^N + g^P\right).\end{aligned}$$

Since $g^{N\times P} = g^N + g^P$, by the uniqueness of the decomposition (1.39), we have: $W^{N\times P} = 0$.

2) The Riemann curvature tensor of the product is:

$$\operatorname{Rm}^{N\times\mathbb{R}} = \frac{1}{2} g^N \odot g^N = \frac{1}{2}\left(g^N - dt^2\right) \odot \left(g^N + dt^2\right),$$

where we used the fact that: $dt^2 \odot dt^2 = 0$. Therefore $W^{N\times\mathbb{R}} = 0$. □

Finally, we state a couple of important theorems concerning locally conformally flat manifolds.

THEOREM 1.35 (Kuiper 1949). *If (M^n, g) is a simply connected, locally conformally flat, closed Riemannian manifold, then (M^n, g) is conformal to the standard sphere S^n.*

THEOREM 1.36 (Schoen-Yau). *If (M^n, g) is a simply connected, locally conformally flat, complete Riemannian manifold, then there exists a one-to-one conformal map of (M^n, g) into the standard sphere S^n.*

When M^n is not simply connected, it is useful to apply the above results to the universal cover $\left(\tilde{M}^n, \tilde{g}\right)$.

EXERCISE 1.37. *Show that if (M_1, g_1) and (M_2, g_2) are Riemannian manifolds, then the product Riemannian manifold $(M_1 \times M_2, g_1 \times g_2)$ satisfies:*

(1)

$$\operatorname{Rm}_{g_1 \times g_2}(X, Y, Z, W) = \operatorname{Rm}_{g_1}(X_1, Y_1, Z_1, W_1) + \operatorname{Rm}_{g_2}(X_2, Y_2, Z_2, W_2)$$

where $X = (X_1, X_2) \in T(M_1 \times M_2)$, etc.

(2)

$$\operatorname{Rc}_{g_1 \times g_2}(X, Y) = \operatorname{Rc}_{g_1}(X_1, Y_1) + \operatorname{Rc}_{g_2}(X_2, Y_2). \tag{1.47}$$

2.3. Cartan structure equations. We shall often find it convenient to compute curvatures in a **moving** (orthonormal) **frame**. The method of moving frames, which we describe below, was primarily developed first by Elie Cartan and then by S.-S. Chern. Let $\{e_i\}_{i=1}^n$ be a local orthonormal frame field in an open set $U \subset M^n$. The dual orthonormal basis (or **coframe field**) $\{\omega^i\}_{i=1}^n$ of T^*M^n is defined by $\omega^i(e_j) = \delta^i_j$ for all $i, j = 1, \ldots, n$. We may write the metric as

$$g = \sum_{i=1}^n \omega^i \otimes \omega^i.$$

The **connection 1-forms** ω_i^j are the components of the Levi-Civita connection with respect to $\{e_i\}_{i=1}^n$:

$$\nabla_X e_i \doteqdot \sum_{j=1}^n \omega_i^j(X)\, e_j, \tag{1.48}$$

for all $i, j = 1, \ldots, n$ and all vector fields X on U. The connection 1-forms are antisymmetric:

$$\omega_i^j = -\omega_j^i$$

since for all X

$$0 \equiv X\langle e_i, e_j\rangle = \langle \nabla_X e_i, e_j\rangle + \langle e_i, \nabla_X e_j\rangle.$$

From $\omega^i(e_j) = \delta^i_j$ and the product rule we see that

$$\nabla_X \omega^i = -\omega_j^i(X)\,\omega^j. \tag{1.49}$$

The **curvature 2-forms** Rm_i^j on U are defined by:

$$\operatorname{Rm}(X, Y)\, e_i \doteqdot \sum_{j=1}^n \operatorname{Rm}_i^j(X, Y)\, e_j$$

so that $\operatorname{Rm}_i^j(X,Y) = \langle \operatorname{Rm}(X,Y)\, e_i, e_j\rangle$.

THEOREM 1.38 (Cartan structure equations). *The* ***first and second Cartan structure equations*** *are:*

$$d\omega^i = \omega^j \wedge \omega_j^i \tag{1.50}$$

$$\operatorname{Rm}_i^j = d\omega_i^j - \omega_i^k \wedge \omega_k^j. \tag{1.51}$$

PROOF. We compute

$$\begin{aligned} d\omega^i(X,Y) &= \left(\nabla_X \omega^i\right)(Y) - \left(\nabla_Y \omega^i\right)(X) \\ &= -\omega_j^i(X)\,\omega^j(Y) + \omega_j^i(Y)\,\omega^j(X) \end{aligned}$$

and (1.50) follows. From (1.7) and

$$\nabla^2 e_i = \left(\nabla \omega_i^k\right) e_k + \omega_i^k \nabla e_k$$

we have

$$\begin{aligned} &\operatorname{Rm}_i^j(X,Y) \\ &= \left\langle \nabla^2_{X,Y} e_i - \nabla^2_{Y,X} e_i, e_j \right\rangle \\ &= \left\langle \left(\nabla_X \omega_i^k\right)(Y)\, e_k + \omega_i^k(Y)\,\nabla_X e_k - \left(\nabla_Y \omega_i^k\right)(X)\, e_k - \omega_i^k(X)\,\nabla_Y e_k, e_j \right\rangle \\ &= d\omega_i^k(X,Y)\,\langle e_k, e_j\rangle + \left(\omega_i^k(Y)\,\omega_k^\ell(X) - \omega_i^k(X)\,\omega_k^\ell(Y)\right)\langle e_\ell, e_j\rangle \end{aligned}$$

and (1.51) follows. □

For a surface M^2, we have

$$d\omega^1 = \omega^2 \wedge \omega_2^1, \quad d\omega^2 = \omega^1 \wedge \omega_1^2,$$

$$\operatorname{Rm}_2^1 = d\omega_2^1.$$

In particular, the Gauss curvature is given by

$$K \doteqdot \langle R(e_1,e_2)\,e_2, e_1\rangle = \operatorname{Rm}_2^1(e_1,e_2) = d\omega_2^1(e_1,e_2).$$

EXERCISE 1.39 (Formula for connection 1-forms). *Show that*

$$d\omega^k(e_i,e_j) = \omega_i^k(e_j) - \omega_j^k(e_i).$$

Using this and the first structure equation (1.50) derive the formula for the connection 1-forms:

$$\omega_i^k(e_j) = \frac{1}{2}\left(d\omega^i(e_j,e_k) + d\omega^j(e_i,e_k) - d\omega^k(e_j,e_i)\right). \tag{1.52}$$

Note the similarity between this and the formula for the Christoffel symbols (1.4).

SOLUTION. We compute

$$\begin{aligned}
& d\omega^i(e_j,e_k)+d\omega^j(e_i,e_k)-d\omega^k(e_j,e_i)\\
&=\left(\omega^\ell\wedge\omega_\ell^i\right)(e_j,e_k)+\left(\omega^\ell\wedge\omega_\ell^j\right)(e_i,e_k)-\left(\omega^\ell\wedge\omega_\ell^k\right)(e_j,e_i)\\
&=\omega_j^i(e_k)-\omega_k^i(e_j)+\omega_i^j(e_k)-\omega_k^j(e_i)-\omega_j^k(e_i)+\omega_i^k(e_j)\\
&=2\omega_i^k(e_j).
\end{aligned}$$

EXERCISE 1.40 (2nd Bianchi). *Prove*

$$(d_\nabla \operatorname{Rm})_i^j \doteqdot d\operatorname{Rm}_i^j-\omega_i^k\wedge\operatorname{Rm}_k^j+\omega_k^j\wedge\operatorname{Rm}_i^k=0.$$

Here $d_\nabla \operatorname{Rm}_i^j$ *is the* ***exterior covariant derivative*** *of* Rm *considered as a 2-form with values in* $T^*M^n\otimes TM^n$. *This is an equivalent formulation of the second Bianchi identity.*

SOLUTION. Using the second structure equations, we compute

$$\begin{aligned}
& d\operatorname{Rm}_i^j-\omega_i^k\wedge\operatorname{Rm}_k^j+\omega_k^j\wedge\operatorname{Rm}_i^k\\
&=-d\omega_i^k\wedge\omega_k^j+\omega_i^k\wedge d\omega_k^j-\omega_i^k\wedge\left(d\omega_k^j-\omega_k^\ell\wedge\omega_\ell^j\right)+\omega_k^j\wedge\left(d\omega_i^k-\omega_i^\ell\wedge\omega_\ell^k\right)\\
&=\omega_i^k\wedge\omega_k^\ell\wedge\omega_\ell^j-\omega_k^j\wedge\omega_i^\ell\wedge\omega_\ell^k=0
\end{aligned}$$

after switching k and ℓ in one of the terms in the last line.

2.4. Moving frame adapted to a hypersurface. The study of hypersurfaces in Riemannian manifolds is useful in probing the geometry of these manifolds. For example, minimal surfaces in 3-manifolds are useful in understanding the topology of the ambient 3-manifold. Let (P^n,g_P) be a Riemannian manifold and D denote the associated covariant derivative (Levi-Civita connection). Given a hypersurface $M^{n-1}\subset P^n$, let $\{e_i\}_{i=1}^n$ be a moving frame in a neighborhood $U\subset P^n$ of a point in M^{n-1}. The connection 1-forms $\left\{\omega_i^j\right\}_{i,j=1}^n$ of (P^n,g_P) satisfy

$$D_Xe_i=\sum_{j=1}^n\omega_i^j(X)\,e_j.$$

Now assume the frame is **adapted** to M^{n-1}, that is, $e_n\doteqdot\nu$ is normal to M^{n-1}. The **second fundamental form** is:

$$h(X,Y)\doteqdot\langle D_X\nu,Y\rangle=\omega_n^j(X)\langle Y,e_j\rangle \tag{1.53}$$

for X and Y tangent to M^{n-1}. The second fundamental form measures the extrinsic geometry of the hypersurface - e.g., how non-parallel the normal is. Let $h_{ij}\doteqdot h(e_i,e_j)=\omega_n^j(e_i)$ so that

$$\omega_n^j=\sum_{i=1}^{n-1}h_{ij}\omega^i.$$

The **mean curvature** is the trace of the second fundamental form:

$$H \doteqdot \sum_{i=1}^{n} h\left(e_i, e_i\right).$$

Let g_M denote the induced Riemannian metric on M^{n-1}. The induced Levi-Civita connection ∇ of g_M satisfies

$$\nabla_X e_i \doteqdot \left(D_X e_i\right)^T = \sum_{j=1}^{n-1} \omega_i^j\left(X\right) e_j.$$

where ^{T} denotes the tangent component of a vector. Thus $\left\{\omega_i^j\right\}_{i,j=1}^{n-1}$ are the connection 1-forms of $\left(M^{n-1}, g_M\right)$. The second structure equation gives us the formula for the curvatures of both the ambient manifold and the hypersurface:

$$\begin{aligned}
\left(\operatorname{Rm}_P\right)_i^j &= d\omega_i^j - \sum_{k=1}^{n} \omega_i^k \wedge \omega_k^j, \quad i, j = 1, \ldots, n, \\
\left(\operatorname{Rm}_M\right)_i^j &= d\omega_i^j - \sum_{k=1}^{n-1} \omega_i^k \wedge \omega_k^j, \quad i, j = 1, \ldots, n-1.
\end{aligned}$$

Thus, for $i, j = 1, \ldots, n-1$, we have

$$\begin{aligned}
\left(\operatorname{Rm}_M\right)_i^j &= \left(\operatorname{Rm}_P\right)_i^j + \omega_i^n \wedge \omega_n^j \\
&= \left(\operatorname{Rm}_P\right)_i^j - h_{ik} h_{j\ell} \omega^k \wedge \omega^\ell.
\end{aligned}$$

Applying this to (e_k, e_ℓ) we obtain the **Gauss equations**

$$\left(R_M\right)_{ijk\ell} = \left(R_P\right)_{ijk\ell} + h_{i\ell} h_{jk} - h_{ik} h_{j\ell}. \tag{1.54}$$

For $j = 1, \ldots, n-1$, we have

$$\left(\operatorname{Rm}_P\right)_n^j = d\omega_n^j - \sum_{k=1}^{n-1} \omega_n^k \wedge \omega_k^j.$$

The $(1,1)$-tensor $W \doteqdot \sum_{j=1}^{n-1} \omega_n^j e_j$ is the **Weingarten map**. Considering W as a 1-form with values in TM^{n-1}, we have

$$d_\nabla W = \sum_{j=1}^{n-1} \left(\operatorname{Rm}_P\right)_n^j e_j$$

which is a 2-form with values in TM^{n-1}.

EXERCISE 1.41 (Codazzi equations). *Show that for X, Y, Z tangent to M^{n-1}*

$$\left(\nabla_X h\right)\left(Y, Z\right) - \left(\nabla_Y h\right)\left(X, Z\right) = \left\langle \operatorname{Rm}_P\left(X, Y\right) Z, \nu\right\rangle.$$

REMARK 1.42. *Moving frames are often useful in computing the connections and curvatures of metrics which possess symmetries, such as rotationally symmetric metrics and homogeneous metrics.*

2.5. The Laplacian. Let Δ denote the **Laplacian** (or **Laplace-Beltrami operator**) **acting on functions**, which is globally defined as the divergence[4] of the gradient and given in local coordinates by:

$$\Delta \doteqdot \operatorname{div} \nabla = g^{ij} \nabla_i \nabla_j = g^{ij} \left(\frac{\partial^2}{\partial x^i \partial x^j} - \Gamma_{ij}^k \frac{\partial}{\partial x^k} \right). \tag{1.55}$$

There are other equivalent ways to define Δ such as

$$\Delta f = \sum_{a=1}^{n} e_a \left(e_a f \right) - \left(\nabla_{e_a} e_a \right) f, \tag{1.56}$$

where $\{e_a\}_{a=1}^n$ is an orthonormal frame.

REMARK 1.43. *In euclidean space* $\Delta = \sum_{i=1}^n \frac{\partial^2}{\partial (x^i)^2}$ *and the heat equation is* $\left(\frac{\partial}{\partial t} - \Delta \right) u = 0$. *Especially since the Ricci flow is like a heat equation, we shall often encounter the Laplacian and heat operator.*

EXERCISE 1.44.

(1) *Show that the above two definitions of* Δ *are the same.* HINT*: show that for any function* f *and vectors* X *and* Y *at a point* p *we have the following formula for the* ***Hessian*** $\nabla\nabla f$ *:*

$$\nabla \nabla f \left(X, Y \right) = X \left(Y f \right) - \left(\nabla_X Y \right) f$$

at p *independent of how one extends* X *and* Y *to a neighborhood of* p*. The Laplacian is the trace of the Hessian.*

(2) *Also show that*

$$\Delta f = \frac{1}{\sqrt{|g|}} \sum_{i,j=1}^{n} \frac{\partial}{\partial x^i} \left(\sqrt{|g|} g^{ij} \frac{\partial f}{\partial x^j} \right) \tag{1.57}$$

where $|g| \doteqdot \det g_{ij}$.

We take this opportunity to define the **divergence** of a $(p, 0)$-tensor as

$$\operatorname{div} (\alpha)_{i_1 \cdots i_{p-1}} \doteqdot g^{jk} \nabla_j \alpha_{k i_1 \cdots i_{p-1}} = \nabla_j \alpha_{j i_1 \cdots i_{p-1}}. \tag{1.58}$$

In particular if X is a 1-form, then

$$\operatorname{div} (X) = g^{ij} \nabla_i X_j.$$

The last equality in (1.58) reflects the convention that we take throughout this book, which is not to bother to raise indices and to sum over repeated indices. The reader may think of this as computing in local coordinates at a point where $g_{ij} = \delta_{ij}$. For instance, $R_{ijk\ell} = R_{ijk}^{\ell}$. The Laplacian acting on functions may be written as $\Delta f = \operatorname{div} (\nabla f)$ for any function f. More

[4]See (1.58).

generally, the (**rough**) **Laplacian operator acting on tensors** is given by

$$\Delta = \operatorname{div} \nabla = \operatorname{trace}_g \nabla^2 = g^{ij} \nabla_i \nabla_j = \nabla_i \nabla_i . \tag{1.59}$$

More explicitly, given an (r,s)-tensor β, $\nabla\nabla\beta$ is an $(r+2,s)$-tensor, which we contract to get

$$\Delta \beta (X_1, \ldots, X_r) = \sum_{a=1}^{n} \nabla\nabla\beta (e_a, e_a, X_1, \ldots, X_r) \in \otimes^s TM$$

for all vectors $X_1, \ldots, X_r$.

A particularly useful identity is the following.

LEMMA 1.45 (Commutator of Δ and ∇ on functions). *For any function f*

$$\Delta \nabla_i f = \nabla_i \Delta f + R_{ij} \nabla_j f. \tag{1.60}$$

PROOF. This follows from

$$\Delta \nabla_i f = \nabla_j \nabla_i \nabla_j f = \nabla_i \nabla_j \nabla_j f - R_{jijk} \nabla_k f.$$

□

EXERCISE 1.46 (Bochner formula for $|\nabla f|^2$). *Suppose M^n is closed. Show that for any C^3 function f*

$$\Delta |\nabla f|^2 = 2 |\nabla_i \nabla_j f|^2 + 2R_{ij} \nabla_i f \nabla_j f + 2\nabla_i f \nabla_i (\Delta f) . \tag{1.61}$$

Conclude from this that if $\mathrm{Rc} \geq 0$, $\Delta f \equiv 0$ *and* $|\nabla f| \equiv 1$, *then* ∇f *is parallel, i.e.,* $\nabla\nabla f \equiv 0$, *and* $\mathrm{Rc}(\nabla f, \nabla f) \equiv 0$. *As we shall see, the distance and Busemann functions satisfy* $|\nabla f| = 1$ *a.e. (see Exercise 1.73).*

SOLUTION. We compute

$$\begin{aligned} \Delta |\nabla f|^2 &= \nabla_i \nabla_i |\nabla_j f|^2 = 2\nabla_i (\nabla_i \nabla_j f \nabla_j f) \\ &= 2 |\nabla_i \nabla_j f|^2 + 2\Delta \nabla_j f \nabla_j f \end{aligned}$$

and obtain (1.61) from (1.60). If $\mathrm{Rc} \geq 0$ and $\Delta f = 0$, then (1.61) implies

$$\Delta |\nabla f|^2 \geq 2 |\nabla\nabla f|^2 + 2 \,\mathrm{Rc}(\nabla f, \nabla f) \geq 2 |\nabla\nabla f|^2 .$$

Integrating this over M we have

$$0 \geq \int_M \left[|\nabla\nabla f|^2 + \mathrm{Rc}(\nabla f, \nabla f) \right] d\mu.$$

Since $|\nabla\nabla f|^2 \geq 0$ and $\mathrm{Rc}(\nabla f, \nabla f) \geq 0$, we conclude $\nabla\nabla f \equiv 0$, and $\mathrm{Rc}(\nabla f, \nabla f) \equiv 0$.

EXERCISE 1.47. *Show that*

$$\Delta |\nabla f| = \frac{1}{|\nabla f|} \left(\nabla f \cdot \nabla (\Delta f) + \mathrm{Rc}(\nabla f, \nabla f) + |\nabla\nabla f|^2 - \left| \left\langle \nabla\nabla f, \frac{\nabla f}{|\nabla f|} \right\rangle \right|^2 \right)$$

wherever $|\nabla f| \neq 0$, *and conclude that if* $\mathrm{Rc} \geq 0$, *then*

$$\Delta |\nabla f| \geq \frac{\nabla f}{|\nabla f|} \cdot \nabla (\Delta f).$$

In particular, if $\Delta f = 0$, *then*

$$\Delta |\nabla f| \geq 0. \tag{1.62}$$

EXERCISE 1.48. *Show that if* $\frac{\partial}{\partial t} g_{ij} = -2R_{ij}$ *(Ricci flow), then*

$$\left(\Delta - \frac{\partial}{\partial t}\right) |\nabla f|^2 = 2 |\nabla_i \nabla_j f|^2 + 2\nabla_i f \nabla_i \left(\left(\Delta - \frac{\partial}{\partial t}\right) f\right).$$

2.6. Integration by parts. Besides pointwise formulas, such as (1.61), we shall find integral identities useful. A basic tool is integration by parts. Recall that Stokes' theorem says that

THEOREM 1.49. *If* α *is an* $(n-1)$*-form on a compact differentiable manifold* M^n *with (possibly empty) boundary* ∂M, *then*

$$\int_M d\alpha = \int_{\partial M} \alpha.$$

The **divergence theorem** says

THEOREM 1.50. *Let* (M, g) *be a compact Riemannian manifold. If* X *is a 1-form, then*

$$\int_{M^n} \operatorname{div}(X)\, d\mu = \int_{\partial M^n} \langle X, \nu \rangle\, d\sigma. \tag{1.63}$$

Here ν *is the unit outward normal,* $d\mu$ *denotes the* ***volume form*** *of* g *(see (2.12) for its formula in local coordinates), and* $d\sigma \doteqdot \iota_\nu(d\mu)$ *is the volume form of the boundary* ∂M^n *with respect to the induced metric.*

PROOF. Define the $(n-1)$-form α by

$$\alpha = \iota_X(d\mu).$$

Using $d^2 = 0$ we compute

$$d\alpha = d \circ \iota_X(d\mu) = (d \circ \iota_X + \iota_X \circ d)(d\mu) = \mathcal{L}_X(d\mu) = \operatorname{div}(X) d\mu,$$

where to obtain the last equality, we may compute in an orthonormal frame $e_1, \ldots, e_n$:

$$\begin{aligned} \mathcal{L}_X(d\mu)(e_1, \ldots, e_n) &= \sum_{i=1}^n d\mu(e_1, \ldots, \nabla_{e_i} X, \ldots, e_n) \\ &= \operatorname{div}(X) d\mu(e_1, \ldots, e_n). \end{aligned}$$

Now Stokes' theorem implies

$$\int_M \operatorname{div}(X) d\mu = \int_M d\alpha = \int_{\partial M} \alpha = \int_{\partial M} \iota_X(d\mu) = \int_{\partial M} X(\nu) d\sigma,$$

and the theorem is proved. $\square$

EXERCISE 1.51. *Derive the following consequences of the divergence theorem.*

(1) *On a closed manifold,* $\int_{M^n} \Delta u d\mu = 0$.
(2) (Green) *On a compact manifold,*

$$\int_{M^n} (u\Delta v - v\Delta u)\, d\mu = \int_{\partial M^n} \left(u\frac{\partial v}{\partial \nu} - v\frac{\partial u}{\partial \nu}\right) d\sigma.$$

In particular, on a closed manifold

$$\int_{M^n} u\Delta v d\mu = \int_{M^n} v\Delta u d\mu.$$

(3) *Show that if* f *is a function and* X *is a 1-form, then*

$$\int_{M^n} f \operatorname{div}(X)\, d\mu = -\int_{M^n} \langle \nabla f, X\rangle\, d\mu + \int_{\partial M^n} f\langle X, \nu\rangle\, d\sigma.$$

COROLLARY 1.52. *Let* (M^n, g) *be a closed Riemannian manifold. If* α *is an* (r,s)*-tensor and* β *is an* $(r-1,s)$*-tensor, then*

$$\int_M \langle \alpha, \nabla\beta\rangle\ dV = \int_M \langle \operatorname{div}(\alpha), \beta\rangle\ dV.$$

PROOF. Let $X_j = \alpha_{ji_2\cdots i_r}^{k_1\cdots k_s}\beta_{i_2\cdots i_r}^{k_1\cdots k_s}$. We compute that

$$\operatorname{div} X = \langle \operatorname{div}(\alpha), \beta\rangle + \langle \alpha, \nabla\beta\rangle\ ,$$

and the result follows from the divergence theorem. □

EXERCISE 1.53. *Show that on a closed manifold*

$$\int_{M^n} |\nabla\nabla f|^2\, d\mu + \int_{M^n} \operatorname{Rc}(\nabla f, \nabla f)\, d\mu = \int_{M^n} (\Delta f)^2\, d\mu. \tag{1.64}$$

Since $|\nabla\nabla f|^2 \geq \frac{1}{n}(\Delta f)^2$, *this implies*

$$\int_{M^n} \operatorname{Rc}(\nabla f, \nabla f)\, d\mu \leq \frac{n-1}{n}\int_{M^n} (\Delta f)^2\, d\mu. \tag{1.65}$$

EXERCISE 1.54 (Lichnerowicz). *Suppose* f *is an eigenfunction of the Laplacian with eigenvalue* λ*:*

$$\Delta f + \lambda f = 0.$$

Use (1.65) to show that if $\operatorname{Rc} \geq (n-1)Kg$, *where* $K > 0$ *is a constant, then*

$$\lambda \geq nK. \tag{1.66}$$

Equality is obtained by linear functions on the sphere of radius $1/\sqrt{K}$.

SOLUTION. If f is an eigenfunction with eigenvalue λ, then by (1.65) we have

$$(n-1)K\int_{M^n}|\nabla f|^2\,d\mu \leq \int_{M^n}\mathrm{Rc}(\nabla f,\nabla f)\,d\mu \leq \frac{n-1}{n}\int_{M^n}(\Delta f)^2\,d\mu = \frac{n-1}{n}\lambda^2\int_{M^n}f^2 d\mu$$

and (1.66) follows from

$$\int_{M^n}|\nabla f|^2\,d\mu = \lambda\int_{M^n}f^2 d\mu > 0.$$

3. Laplacian and Hessian comparison theorems

Two fundamental results in Riemannian geometry are the **Laplacian and Hessian comparison theorems for the distance function**. They are directly related to the volume comparison theorem and a special case of the Rauch comparison theorem. The Hessian comparison theorem may also be used to prove the Toponogov triangle comparison theorem (see §6.6.1). The ideas behind these elementary results have a profound influence on geometric analysis and Ricci flow (see especially Volume 2.) Given a point $p \in M^n$, the **exponential map** $\exp_p : T_pM^n \to M^n$ is defined by

$$\exp_p(V) \doteqdot \gamma_V(1)$$

where $\gamma_V : [0,\infty) \to M^n$ is the geodesic emanating from p with $\dot{\gamma}_V(0) = V$. Note that $\gamma_V(r) = \exp_p(rV)$. The distance function is defined by

$$d_p(x) \doteqdot d(x,p).$$

From the triangle inequality, it is easy to see that d_p is a Lipschitz function with Lipschitz constant equal to 1:

$$|d_p(x) - d_p(y)| \leq d(x,y).$$

Recall also that a **Jacobi field** J is a variation of geodesics and satisfies the **Jacobi equation**:

$$\boxed{\nabla_S\nabla_S J + R(J,S)S = 0}$$

which is derived immediately from the geodesic equation $\nabla_S S \equiv 0$ since

$$0 = \nabla_J\nabla_S S = \nabla_S\nabla_J S + R(J,S)S$$

and $[J,S] = 0$. Given $p \in M^n$ and $V, W \in T_pM$, define a 1-parameter family of geodesics

$$\gamma_s : [0,\infty) \to M^n$$

by

$$\gamma_s(r) \doteqdot \exp_p(r(V+sW)) = \gamma_{V+sW}(r).$$

We may define a Jacobi field J_W along $\gamma_0 = \gamma_V$ by

$$J_W(r) \doteqdot \left.\frac{\partial}{\partial s}\right|_{s=0} \gamma_{V+sW}(r).$$

Given a point $p \in M^n$ and a unit speed geodesic $\gamma : [0, \infty) \to M^n$ with $\gamma(0) = p$, either γ is a ray (i.e., minimal on each finite subinterval) or there exists a unique $r_\gamma \in (0, \infty)$ such that $d(\gamma(r), p) = r$ for $r \leq r_\gamma$ and $d(\gamma(r), p) < r$ for $r > r_\gamma$. We say that $\gamma(r_\gamma)$ is a **cut point to** p **along** γ. The **cut locus** $\mathrm{Cut}(p)$ of p in M^n is the set of all cut points of p. Now let

$$D_p \doteqdot \left\{ V \in T_pM^n : d\left(\exp_p(V), p\right) = |V| \right\}, \tag{1.67}$$

which is a closed subset of T_pM^n. We define $C_p \doteqdot \partial D_p$ to be the cut locus of p in the tangent space. We have $\mathrm{Cut}(p) = \exp_p(C_p)$. We have $\exp_p :$ $\mathrm{int}(D_p) \to M^n - \mathrm{Cut}(p)$ is a diffeomorphism. (We call $\mathrm{int}(D_p)$ the interior to the cut locus in the tangent space T_pM.)

DEFINITION 1.55. *A point $x \in M$ is a **conjugate point** of $p \in M^n$ if x is a singular value of $\exp_p : T_pM \to M$. That is, $x = \exp_p(V)$, for some $V \in T_pM$, where $\left(\exp_p\right)_* : T_V(T_pM) \to T_{\exp_p(V)}M$ is singular.*

Equivalently, $\gamma(r)$ is a conjugate point to p along γ if there is a nontrivial Jacobi field along γ vanishing at the endpoints. Note that it follows that $r \geq r_\gamma$.

LEMMA 1.56. *A point $\gamma(r)$ is a cut point to p along γ if and only if r is the smallest positive number such that either $\gamma(r)$ is a conjugate point to p along γ or there exist two distinct minimal geodesics joining p and $\gamma(r)$.*

Given $V \in T_pM^n$, let $\gamma_V : [0, \infty) \to M^n$ denote the constant speed geodesic with $\dot\gamma_V(0) = V$, i.e., $\gamma_V(r) \doteqdot \exp_p(rV)$. For each unit vector $V \in T_pM^n$ there exists at most a unique $r_V \in (0, \infty)$ such that $\gamma_V(r_V)$ is a cut point of p along γ_V. Furthermore, if we set $r_V \doteqdot \infty$ when γ_V is a ray, then the map from the unit tangent space at p to $(0, \infty]$ given by $V \mapsto r_V$ is a continuous function (see §11.6 of [**50**] for example). Hence we have

$$C_p = \partial D_p = \{r_V V : V \in T_pM^n,\ |V| = 1,\ \gamma_V \text{ is not a ray}\}$$

has measure zero with respect to the euclidean measure on $(T_pM^n, g(p))$.[5] Since $\exp_p$ is a smooth function, we conclude that

LEMMA 1.57. $\mathrm{Cut}(p) = \exp_p(C_p)$ *has measure zero with respect to the **Riemannian measure** on (M^n, g).*[6]

Now if $x \notin \mathrm{Cut}(p)$, then d_p is smooth at x and $|\nabla d_p(x)| = 1$ by (1.76). Since $\mathrm{Cut}(p)$ has measure zero, we have $|\nabla d_p| = 1$ a.e. on M^n.

An alternate proof of Lemma 1.57 can be given as follows. Since d_p is locally Lipschitz, we have d_p is C^1 a.e. On the other hand it is easy

[5]For C_p is the radial graph of a continuous function.

[6]See [**438**], section 2.5 for a discussion of Riemannian measure.

to see that d_p is not C^1 at those points x in $\mathrm{Cut}(p)$ for which there are two distinct minimal geodesics joining p to x. Thus, this set of points in $\mathrm{Cut}(p)$ has measure zero. We also know that by Sard's theorem, the points in $\mathrm{Cut}(p)$ which are conjugate points form a measure zero set (since these points are singular values of $\exp_p$). We conclude that $\mathrm{Cut}(p)$ has measure zero.

DEFINITION 1.58. *The* ***injectivity radius*** $\mathrm{inj}(p)$ *of a point* $p \in M^n$ *is defined to be the supremum of all* $r > 0$ *such that* $\exp_p$ *is an embedding when restricted to* $B\left(\vec{0}, r\right)$. *Equivalently,*

(1) $\mathrm{inj}(p)$ *is the distance from* $\vec{0}$ *to* C_p *with respect to* $g(p)$,
(2) $\mathrm{inj}(p)$ *is the Riemannian distance from* p *to* $\mathrm{Cut}(p)$.

The injectivity radius of a Riemannian manifold is defined to be

$$\mathrm{inj}(M^n, g) \doteqdot \inf\{\mathrm{inj}(p) : p \in M^n\}.$$

When M^n is compact, the injectivity radius is always positive.

3.1. Laplacian comparison theorem. The idea of comparison theorems is to compare a geometric quantity on a Riemannian manifold with the corresponding quantity on a model space. Typically, in Riemannian geometry, model spaces have constant sectional curvature. As we shall see later, model spaces for Ricci flow are gradient Ricci solitons.

THEOREM 1.59 (Laplacian Comparison). *If* (M^n, g) *is a complete Riemannian manifold with* $\mathrm{Rc} \geq -(n-1)H$, *where* $H > 0$, *and if* $p \in M^n$, *then for any* $x \in M^n$ *where* $d_p(x)$ *is smooth, we have*

$$\boxed{\Delta d_p(x) \leq (n-1)\sqrt{H}\coth\left(\sqrt{H} d_p(x)\right).} \tag{1.68}$$

On the whole manifold, the Laplacian comparison theorem (1.68) holds in the ***sense of distributions***. *That is, for any nonnegative* C^∞ *function* φ *on* M^n *with compact support, we have*

$$\int_{M^n} d_p \Delta\varphi d\mu \leq \int_{M^n} (n-1)\sqrt{H}\coth\left(\sqrt{H} d_p\right)\varphi d\mu.$$

See section 4 for the proof. From Theorem 1.59 we can derive the following.

COROLLARY 1.60. *If* $H \geq 0$, *then*

$$\Delta d_p \leq \frac{n-1}{d_p} + (n-1)\sqrt{H} \tag{1.69}$$

in the sense of distributions. That is, for any nonnegative $\varphi \in C^\infty(M^n)$ *with compact support,*

$$\int_{M^n} d_p \Delta\varphi d\mu \leq \int_{M^n} \left(\frac{n-1}{d_p} + (n-1)\sqrt{H}\right)\varphi d\mu.$$

In particular, if (M^n, g) *is a complete Riemannian manifold with* $\mathrm{Rc} \geq 0$, *then for any* $p \in M^n$

$$\Delta d_p \leq \frac{n-1}{d_p} \tag{1.70}$$

in the sense of distributions.

REMARK 1.61. *A statement analogous to (1.68) holds when the Ricci curvature is bounded from below by a positive constant. Estimate (1.68) is sharp as can be seen from considering space forms of constant curvature* $-H$. *If* $H = 0$, *then (1.70) is sharp since on euclidean space* $\Delta |x| = \frac{n-1}{|x|}$.

A consequence of the Laplacian comparison theorem is (see the next section for the proof):

THEOREM 1.62 (Bishop Volume Comparison). *If* (M^n, g) *is a complete Riemannian manifold with* $\mathrm{Rc} \geq (n-1)K$, *then for any* $p \in M^n$, *the volume ratio*

$$\frac{\mathrm{Vol}(B(p,r))}{\mathrm{Vol}_K(B(p_K,r))}$$

is a nonincreasing function of r, *where* p_K *is a point in the* n*-dimensional simply connected space form of constant curvature* K *and* Vol_K *denotes the volume in the space form. In particular*

$$\mathrm{Vol}(B(p,r)) \leq \mathrm{Vol}_K(B(p_K,r)) \tag{1.71}$$

for all $r > 0$. *Given* p *and* $r > 0$, *equality holds in (1.71) if and only if* $B(p,r)$ *is isometric to* $B(p_K, r)$.

In the case of nonnegative Ricci curvature we have the following.

COROLLARY 1.63. *If* (M^n, g) *is a complete Riemannian manifold with* $\mathrm{Rc} \geq 0$, *then for any* $p \in M^n$, *the volume ratio* $\frac{\mathrm{Vol}(B(p,r))}{r^n}$ *is a nonincreasing function of* r. *Since* $\lim_{r\to 0} \frac{\mathrm{Vol}(B(p,r))}{r^n} = \omega_n$, *we have* $\frac{\mathrm{Vol}(B(p,r))}{r^n} \leq \omega_n$ *for all* $r > 0$, *where* ω_n *is the volume of the euclidean unit* n*-ball.*

One of the many useful consequences of this is the following *characterization of euclidean space.*

COROLLARY 1.64 (Volume characterization of $\mathbb{R}^n$). *If* (M^n, g) *is a complete noncompact Riemannian manifold with* $\mathrm{Rc} \geq 0$ *and if for some* $p \in M^n$

$$\lim_{r\to\infty} \frac{\mathrm{Vol}(B(p,r))}{r^n} = \omega_n,$$

then (M^n, g) *is isometric to euclidean space.*

PROOF. By the Bishop volume comparison theorem, we have $\frac{\mathrm{Vol}(B(p,r))}{r^n} \equiv \omega_n$ for all $r > 0$. The result now follows from the equality case. □

The Bishop Volume Comparison Theorem has been generalized to the Bishop-Gromov relative volume comparison theorem (we follow the presentation given by S. Zhu on pp. 226-8 of [**538**], see also [**340**]). Let (M^n, g) be a complete Riemannian manifold and $p \in M^n$. Given a measurable subset Γ of the unit sphere $S_p^{n-1} \subset T_pM$ and $0 < r \leq R < \infty$, define the annular type region:

$$A_{r,R}^{\Gamma}(p) \doteqdot \left\{ x \in M^n : \begin{array}{l} r \leq d(x,p) \leq R \text{ \& there exists a unit speed minimal} \\ \text{geodesic } \gamma \text{ from } \gamma(0) = p \text{ to } x \text{ satisfying } \gamma'(0) \in \Gamma \end{array} \right\}$$
$$\subset B(p,R) \backslash B(p,r).$$

Note that if $\Gamma = S_p^{n-1}$, then $A_{r,R}^{\Gamma}(p) = B(p,R) \backslash B(p,r)$. Given $H \in \mathbb{R}$ and a point p_H in the n-dimensional simply connected space form of constant curvature H, let $A_{r,R}^{\Gamma}(p_H)$ denote the corresponding set in the space form.

THEOREM 1.65. *Let (M^n, g) be a complete Riemannian manifold with $\operatorname{Rc}(g) \geq (n-1)Hg$. If $0 \leq r \leq R \leq S$, $r \leq s \leq S$ and if $\Gamma \subset S_p^{n-1}$ is a measurable subset, then*

$$\frac{\operatorname{Vol}\left(A_{s,S}^{\Gamma}(p)\right)}{\operatorname{Vol}^H\left(A_{s,S}^{\Gamma}(p_H)\right)} \leq \frac{\operatorname{Vol}\left(A_{r,R}^{\Gamma}(p)\right)}{\operatorname{Vol}^H\left(A_{r,R}^{\Gamma}(p_H)\right)},$$

where Vol^H *denotes the volume in the space form.*

Taking $r = s = 0$ and $\Gamma = S_p^{n-1}$ yields:

COROLLARY 1.66 (Bishop-Gromov relative volume comparison theorem). *If $0 < R \leq S$, then*

$$\frac{\operatorname{Vol}(B(p,S))}{\operatorname{Vol}^H(B(p_H,S))} \leq \frac{\operatorname{Vol}(B(p,R))}{\operatorname{Vol}^H(B(p_H,R))}.$$

Then taking the limit as $R \to 0$ gives:

COROLLARY 1.67. *If $S \geq 0$, then*

$$\operatorname{Vol}(B(p,S)) \leq \operatorname{Vol}^H(B(p_H,S)).$$

As a consequence, we have the following result about the volume growth of a complete noncompact manifold with nonnegative Ricci curvature.

COROLLARY 1.68 (Yau - $\operatorname{Rc} \geq 0$ has at least linear volume growth). *Let (M^n, g) be a complete noncompact Riemannian manifold with nonnegative Ricci curvature. For any point $p \in M^n$, there exists a constant $C > 0$ such that for any $r \geq 1$*

$$\operatorname{Vol}(B(p,r)) \geq Cr.$$

PROOF. Let $x \in M^n$ be a point with $d(x,p) = r \geq 2$. By the Bishop-Gromov relative volume comparison theorem, we have

$$\frac{\operatorname{Vol}(B(x,r+1)) - \operatorname{Vol}(B(x,r-1))}{\operatorname{Vol}(B(x,r-1))} \leq \frac{(r+1)^n - (r-1)^n}{(r-1)^n} \leq \frac{C(n)}{r}. \tag{1.72}$$

Since $B(p,1) \subset B(x,r+1) \backslash B(x,r-1)$ and $B(x,r-1) \subset B(p,2r-1)$ by (1.72) we have

$$\operatorname{Vol}(B(p,2r-1)) \geq \operatorname{Vol}(B(x,r-1)) \geq \frac{\operatorname{Vol}(B(p,1))}{C(n)} r.$$

We have proved the corollary for $r \geq 3$. Clearly it is then true for any $r \geq 1$ (or any other positive constant). □

EXAMPLE 1.69. *A simple example of a complete manifold with nonnegative sectional curvature and linear volume growth is* $S^{n-1} \times \mathbb{R}$ *(we may replace* S^{n-1} *by any closed manifold with nonnegative sectional curvature). If we want* M^n *to also have positive sectional curvature at least at one point, then we may take a cylinder* $S^{n-1} \times [0,\infty)$, *attach a hemispherical cap, and then smooth out the metric (see also Example 5.16)*

3.2. Cheeger-Gromoll splitting theorem and manifolds of nonnegative curvature. In the study of manifolds with nonnegative curvature, often (especially when the curvature is not strictly positive) the manifolds split as the product of a lower dimensional manifold with a line. Recall that a **geodesic line** is a unit speed geodesic $\gamma : (-\infty,\infty) \to M^n$ such that the distance between any points on γ is the length of the arc of γ between those two points; that is, for any $s_1, s_2 \in (-\infty,\infty)$, $d(\gamma(s_1), \gamma(s_2)) = |s_2 - s_1|$. Similarly, a unit speed geodesic $\beta : [0,\infty) \to M^n$ is a **geodesic ray** if satisfies the same condition as above. Given a ray $\beta : [0,\infty) \to M^n$, the **Busemann function**

$$b_\beta : M^n \to \mathbb{R}$$

associated to β is defined by

$$b_\beta(x) \doteqdot \lim_{s\to\infty} (s - d(\beta(s), x)). \tag{1.73}$$

In euclidean space the Busemann function is linear:

EXERCISE 1.70. *Let* (M^n, g) *be euclidean space. Show that for any unit vector* $V \in \mathbb{R}^n$, *the Busemann function* b_{γ_V} *associated to the geodesic ray* $\gamma_V : [0,\infty) \to \mathbb{R}^n$ *defined by* $\gamma_V(s) \doteqdot sV$ *is the linear function given by*

$$b_{\gamma_V}(x) = \langle x, V\rangle$$

for all $x \in \mathbb{R}^n$.

The Busemann function is well-defined and finite because of the following.

EXERCISE 1.71. *Show that given* $x \in M^n$, *the function* $s \mapsto s - d(\beta(s), x)$ *is nondecreasing and bounded above by* $d(x, \beta(0))$.

Just like the distance function to a point, we have:

EXERCISE 1.72 (Busemann function is Lipschitz). *Show that for any Riemannian manifold* (M^n, g) *and geodesic ray* β, *the Busemann function* b_β *satisfies*

$$|b_\beta(x) - b_\beta(y)| \leq d(x,y)$$

for all $x, y \in M^n$*. I.e.,* b_β *is Lipschitz with Lipschitz constant* 1*. Note that by Rademacher's theorem,* b_β *is* C^1 *a.e.*

EXERCISE 1.73. *Show that* $|\nabla b_\beta| = 1$ *at points where it is* C^1*.*

More generally, a Lipschitz function f is called a **distance function** if $|\nabla f| = 1$ where it is C^1; e.g., $|\nabla f| = 1$ a.e. The Busemann function associated to a ray β may be thought of as the renormalized distance function to the point at infinity determined by the ray.

EXERCISE 1.74 ($\mathrm{Rc} \geq 0$ implies Busemann function is subharmonic). *Use Corollary 1.60 to show that if* β *is a geodesic ray in a Riemannian manifold with* $\mathrm{Rc} \geq 0$*, then* $\Delta b_\beta \geq 0$ *in the sense of distributions.*

For the proof of the following, see section 4.5.

PROPOSITION 1.75 (Mean Value Inequality for $\mathrm{Rc} \geq 0$). *If* (M^n, g) *is a complete Riemannian manifold with* $\mathrm{Rc} \geq 0$ *and if* f *is a Lipschitz function bounded from above and with* $\Delta f \geq 0$ *in the sense of distributions (**subharmonic**), then for any* $x \in M^n$ *and* $0 < r < \mathrm{inj}(x)$

$$f(x) \leq \frac{1}{\omega_n r^n} \int_{B(x,r)} f \, d\mu$$

where ω_n *is the volume of the unit euclidean* n*-ball.*

In the case where the sectional curvature is bounded from above we have the following (see p. 75 of [**447**]).

PROPOSITION 1.76 (Mean Value Inequality for $\mathrm{sect} \leq H$). *Suppose that* (M^n, g) *is a complete Riemannian manifold with* $\mathrm{sect}(g) \leq H$ *in a ball* $B(x, r)$ *where* $r < \mathrm{inj}(g)$*. If* $f \in C^\infty(M^n)$ *is subharmonic:* $\Delta f \geq 0$ *and if* f *is bounded from below on* M^n*, then*

$$f(x) \leq \frac{1}{V_H(r)} \int_{B(x,r)} f \, d\mu$$

where $V_H(r)$ *is the volume of a ball of radius* r *in the complete simply connected manifold of constant sectional curvature* H*.*

Another fundamental consequence of Corollary 1.60 is the following (see [**104**], [**105**]).

THEOREM 1.77 (Cheeger-Gromoll 1971). *Suppose* (M^n, g) *is a complete Riemannian manifold with* $\mathrm{Rc} \geq 0$ *and suppose there is a geodesic line in* M^n*. Then* (M^n, g) *is isometric to* $\mathbb{R} \times \left(N^{n-1}, h\right)$ *with the product metric, where* $\left(N^{n-1}, h\right)$ *is a Riemannian manifold.*

The proof goes follows (we leave it to the reader to fill in any missing details; see also [**447**]). Given a geodesic line γ, consider the two Busemann functions $b_{\gamma_\pm}$ associated to the geodesic rays $\gamma_\pm : [0, \infty) \to M^n$ defined by $\gamma_\pm(s) = \gamma(\pm s)$ for $s \geq 0$. We have $\Delta b_{\gamma_\pm} \geq 0$ in the sense of distributions and hence $\Delta\left(b_{\gamma_+}+b_{\gamma_-}\right)\geq 0$. It is also easy to see that $\left(b_{\gamma_+}+b_{\gamma_-}\right)(x)=0$ for $x\in\gamma$, and from $d\left(\gamma\left(s\right),\gamma\left(-s\right)\right)=2s$ that $b_{\gamma_+}+b_{\gamma_-}\leq 0$ on M^n. Hence we may apply the Mean Value Inequality on small balls centered at points on γ to conclude that $b_{\gamma_+}+b_{\gamma_-}\equiv 0$ in a neighborhood of γ. By applying the Mean Value Inequality again, we see that the set of points in M^n where $b_{\gamma_+}+b_{\gamma_-}=0$ is open. Since the set is also closed and nonempty, we have $b_{\gamma_+}+b_{\gamma_-}\equiv 0$ on M^n and hence also $\Delta\left(b_{\gamma_+}+b_{\gamma_-}\right)\equiv 0$. Since $\Delta b_{\gamma_\pm}\geq 0$, this implies $\Delta b_{\gamma_\pm}=0$ in the sense of distributions. Standard regularity theory now implies $b_{\gamma_\pm}$ is smooth. Hence $\left|\nabla b_{\gamma_\pm}\right|\equiv 1$. But now Exercise 1.46 implies $\nabla b_{\gamma_\pm}$ is a nonzero parallel gradient vector field on M^n. By the deRham theorem, this implies (M^n,g) is isometric to $\mathbb{R}\times\left(N^{n-1},h\right)$, where $\left(N^{n-1},h\right)=\left\{x\in M^n : b_{\gamma_+}(x)=0\right\}$.

REMARK 1.78. *In the study of the Ricci flow on* 3*-manifolds one of the primary singularity models is the round cylinder* $S^2\times\mathbb{R}$. *This singularity model corresponds to neck pinching.*

A submanifold $S\subset M^n$ is **totally convex** if for every $x,y\in S$ and any geodesic γ (not necessarily minimal) joining x and y we have $\gamma\subset S$. We say that S is **totally geodesic** if its second fundamental form is zero. In particular, a path in S is a geodesic in S if and only if it is a geodesic in M^n.

Given a noncompact manifold (M^n,g) we say that a submanifold is a **soul** if it is a closed, totally convex, totally geodesic submanifold such that M^n is diffeomorphic to its normal bundle.

Generalizing earlier work of Gromoll-Meyer 1969 [**240**], Cheeger-Gromoll 1972 [**105**] (see also Poor 1974 [**429**]) proved the following.

THEOREM 1.79 (Soul). *Let (M^n,g) be a complete Riemannian manifold with nonnegative sectional curvature. Then there exists a soul. If the sectional curvature is positive, then the soul is a point (e.g., M^n is diffeomorphic to $\mathbb{R}^n$.)*

Furthermore, Sharafutdinov [**459**] proved that any two souls are isometric. In 1994 Perelman [**414**] proved the following result.

THEOREM 1.80 (Soul Conjecture). *If (M^n,g) is a complete Riemannian manifold with nonnegative sectional curvature everywhere and positive sectional curvature at some point, then the soul is a point.*

EXERCISE 1.81. *Show that if (M^n,g) is a complete Riemannian manifold with nonnegative sectional curvature and β is a ray, then the Busemann function b_β associated to β is convex.*

SOLUTION. Given $s\geq 0$, let $r(x)\doteqdot d\left(\beta(s),x\right)$. By (1.110), we have $\nabla_i\nabla_j r\leq\frac{1}{r}g_{ij}$ in the C^2 sense wherever r is smooth and in the sense of support functions where r is not C^∞. From taking the limit as $s\to\infty$ in the definition (1.73), we would guess $\nabla_i\nabla_j b_\beta\geq 0$. To prove this rigorously, see [**234**] for example.

An important tool in the study of manifolds with nonnegative curvature is the **Sharafutdinov retraction** (see the above references).

3.3. Hessian comparison theorem. The following roughly says that the larger the curvature, the smaller the Hessian of the distance function.

PROPOSITION 1.82 (Hessian comparison theorem). *Let* $i = 1, 2$. *Let* (M_i^n, g_i) *be complete Riemannian n-manifolds, let* $\gamma_i : [0, L] \to M_i^n$ *be geodesics parametrized by arc length such that* γ_i *does not intersect the cut locus of* $\gamma_i(0)$, *and let* $d_i \doteqdot d(\cdot, \gamma_i(0))$. *If for all* $t \in [0, L]$ *we have*

$$K_{g_1}(V_1 \wedge \dot{\gamma}_1(t)) \geq K_{g_2}(V_2 \wedge \dot{\gamma}_2(t))$$

for all unit vectors $V_i \in T_{\gamma_i(t)} M_i^n$ *perpendicular to* $\dot{\gamma}_i(t)$, *then*

$$\nabla\nabla d_1(X_1, X_1) \leq \nabla\nabla d_2(X_2, X_2)$$

for all $X_i \in T_{\gamma_i(t)} M_i^n$ *perpendicular to* $\dot{\gamma}_i(t)$ *and* $t \in (0, L]$.

See (1.110) at the end of section 4 for the proof of a special case of this.

4. Geodesic polar coordinates

4.1. Exponential map and geodesic coordinate expansion of the metric and volume form. In this section we give the proofs of the Laplacian and Hessian comparison theorems for the distance function and the corresponding volume and Rauch comparison theorems. The comparison geometry viewpoint we take here is foundational for our exposition in Volume 2 of Perelman's $\mathcal{L}$-function for the Ricci flow. Recall that the exponential map $\exp_p : T_p M^n \to M^n$ is defined by $\exp_p(V) \doteqdot \gamma_V(1)$ where $\gamma_V : [0, \infty) \to M^n$ is the geodesic emanating from p with $\dot{\gamma}_V(0) = V$. Given an orthonormal frame $\{e_i\}_{i=1}^n$ at p, let $\left\{X^i\right\}_{i=1}^n$ denote the standard euclidean coordinates on $T_p M$ defined by $V \doteqdot \sum_{i=1}^n V^i e_i$. **Geodesic coordinates** are defined by

$$x^i \doteqdot X^i \circ \exp_p^{-1} : M^n - \mathrm{Cut}(p) \to \mathbb{R}.$$

In geodesic coordinates, we have (see Lemma 3.4 on p. 210 of [**447**])

$$\begin{aligned} g_{ij} = \delta_{ij} &- \frac{1}{3} R_{ipqj} x^p x^q - \frac{1}{6} \nabla_r R_{ipqj} x^p x^q x^r \\ &+ \left(-\frac{1}{20} \nabla_r \nabla_s R_{ipqj} + \frac{2}{45} R_{ipqm} R_{jrsm} \right) x^p x^q x^r x^s + O\left(r^5\right) \end{aligned}$$

so that $g_{ij} = \delta_{ij} + O\left(r^2\right)$, and

(1.74)

$$\begin{aligned} \det g_{ij} = 1 &- \frac{1}{3} R_{ij} x^i x^j - \frac{1}{6} \nabla_k R_{ij} x^i x^j x^k \\ &- \left(\frac{1}{20} \nabla_\ell \nabla_k R_{ij} + \frac{1}{90} R_{pijq} R_{pk\ell q} - \frac{1}{18} R_{ij} R_{k\ell} \right) x^i x^j x^k x^\ell + O\left(r^5\right). \end{aligned}$$

This formula exhibits how the curvature and its derivatives affects the **volume form**

$$d\mu \doteqdot \sqrt{\det g_{ij}}\ dx^1 \wedge \cdots \wedge dx^n$$

where $\{x^i\}$ is a positively oriented local coordinate system. In particular we see that the leading order term on the RHS of (1.74) is expressed in terms of the Ricci curvature and that positive Ricci curvature yields a negative contribution. Next we shall see the more precise effect of a lower bound on the Ricci curvature.

EXERCISE 1.83 (Expansion for volumes of balls). *Show that*

$$\operatorname{Vol}(B(p,r)) = \omega_n r^n \left(1 - \frac{R(p)}{6(n+2)} r^2 + O\left(r^3\right)\right).$$

(Compare with [**218**].*)*

HINT. Show that (1.74) implies

$$\sqrt{\det g_{ij}(x)} = 1 - \frac{1}{6} R_{ij}(p)\, x^i x^j + O\left(|x|^3\right),$$

where $|x| = d(x,p)$ and we abused notation by letting x, x^i denote both the point and the coordinates and Exercise 1.12.

EXERCISE 1.84. *Show that in geodesic coordinates centered at a point* $p \in M$, *we have* $g_{ij}(p) = \delta_{ij}$ *and* $\frac{\partial}{\partial x^i} g_{jk}(p) = 0$.

HINT. One method is as follows. Show that $\Gamma_{ij}^k(p)\, V^i V^j = 0$ for all $V \in T_p M$ and then use the identity

$$\frac{\partial}{\partial x^i} g_{jk} = \Gamma_{ij}^\ell g_{\ell k} + \Gamma_{ik}^\ell g_{\ell j}.$$

4.2. Geodesic polar coordinates and the Jacobian. Typically we say that the geometry is *controlled* if there is a curvature bound and an injectivity radius lower bound. Since in the presence of a curvature bound, a lower bound on the volume gives a lower bound on the injectivity radius (see Exercise **??**), we are interested in the volume of balls. To understand the volumes of balls and their boundary spheres, it convenient to consider geodesic polar coordinates. Given a point $p \in M^n$, let $\{x^i\}_{i=1}^n$ be local polar coordinates on $T_p M^n - \{p\}$. That is,

$$x^n(v) = r(v) = |v|, \quad \text{and } x^i(v) = \theta^i\left(\frac{v}{|v|}\right) \text{ for } 1 \leq i \leq n-1,$$

where $\{\theta^i\}_{i=1}^{n-1}$ are local coordinates on $S_p^{n-1} \doteqdot \{v \in T_p M^n : |v| = 1\}$. Let $\exp_p : T_p M^n \to M^n$ be the exponential map. We call the coordinate system

$$x = \left\{x^i \circ \exp_p^{-1}\right\} : B(p, \operatorname{inj}(p)) - \{p\} \to \mathbb{R}^n$$

a **geodesic polar coordinate system**.[7] Abusing notation we let

$$r \doteqdot x^n \circ \exp_p^{-1}, \quad \text{and } \theta^i \doteqdot x^i \circ \exp_p^{-1}$$

for $i = 1, \ldots, n-1$, so that

$$\frac{\partial}{\partial r} = \left(x^{-1}\right)_* \frac{\partial}{\partial x^n}, \quad \text{and } \frac{\partial}{\partial \theta^i} = \left(x^{-1}\right)_* \frac{\partial}{\partial x^i},$$

which form a basis of vector fields on $B(p, \operatorname{inj}(p)) - \{p\}$. The **Gauss Lemma** says that

$$\operatorname{grad} r = \frac{\partial}{\partial r} \tag{1.75}$$

at all points outside the cut locus of p, so that

$$|\operatorname{grad} r|^2 = \left|\frac{\partial}{\partial r}\right|^2 = \left\langle \operatorname{grad} r, \frac{\partial}{\partial r} \right\rangle = \frac{\partial r}{\partial r} = 1 \tag{1.76}$$

and

$$g_{in} \doteqdot g\left(\frac{\partial}{\partial r}, \frac{\partial}{\partial \theta^i}\right) = \frac{\partial r}{\partial \theta^i} = 0$$

for $i = 1, \ldots, n-1$. We may then write the metric as

$$g = dr \otimes dr + g_{ij} d\theta^i \otimes d\theta^j,$$

where $g_{ij} \doteqdot g\left(\frac{\partial}{\partial \theta^i}, \frac{\partial}{\partial \theta^j}\right)$. Along each geodesic ray emanating from p,

$$\frac{\partial}{\partial \theta^i} \text{ is a Jacobi field} \tag{1.77}$$

before the first conjugate point for each $i \leq n-1$. We call

$$\boxed{\mathrm{J} \doteqdot \sqrt{\det g_{ij}} = \sqrt{\det \left\langle \tfrac{\partial}{\partial \theta^i}, \tfrac{\partial}{\partial \theta^j} \right\rangle_g}} \tag{1.78}$$

the **Jacobian of the exponential map**. The Jacobian of the exponential map is the volume density in polar coordinates. The volume form of g is:

$$d\mu = \sqrt{\det g_{ij}}\, d\theta^1 \wedge \cdots \wedge d\theta^{n-1} \wedge dr = \mathrm{J}\, d\Theta \wedge dr$$

in a positively oriented polar coordinate system, where

$$d\Theta \doteqdot d\theta^1 \wedge \cdots \wedge d\theta^{n-1}. \tag{1.79}$$

Note that by the Gauss Lemma, we have $g_{nn} = 1$ and $g_{in} = 0$ for $i \leq n-1$, so that $\det (g_{ij})_{i,j=1}^n = \det (g_{ij})_{i,j=1}^{n-1}$. If $\gamma(\bar{r})$ is a conjugate point to p along γ, then $\mathrm{J}(r) \to 0$ as $r \to \bar{r}$.

EXERCISE 1.85. *Suppose along a geodesic ray emanating from p we have*

$$\nabla_{\frac{\partial}{\partial r}} \frac{\partial}{\partial \theta^i}(p) = \lim_{r \to 0} \left(\frac{1}{r} \frac{\partial}{\partial \theta^i}\right) \doteqdot E_i \in T_p M^n \tag{1.80}$$

[7] Such coordinates are also called **geodesic spherical coordinates**.

exists and is orthonormal (we shall often assume this in the sequel). Show that

$$\boxed{\lim_{r\to 0}\frac{\mathrm{J}}{r^{n-1}}=1.} \tag{1.81}$$

Note that in this case $d\Theta = d\sigma_{S^{n-1}}$ is the volume form of the unit $(n-1)$-sphere. It is convenient to assume the normalization (1.81) when considering the Jacobian. We shall explicitly say this when we do this.

One way to convince oneself of (1.81) is note that (M^n, cg, p) converges as $c\to\infty$ in the pointed limit $\left(\mathbb{R}^n,\vec{0}\right)$, so that the limit in (1.81) should equal the euclidean value.

Exercise 1.86. *Show that the Gauss Lemma is equivalent to the following statement. If $p\in M$, $V\in T_pM$, and $W\in T_{tV}\left(T_pM\right)\cong T_pM$ are such that $\langle W,V\rangle=0$, then*

$$\left\langle \left(\exp_p\right)_*\left(W_{tV}\right),\left(\exp_p\right)_*\left(V_{tV}\right)\right\rangle=0.$$

4.3. The second fundamental form of the distance spheres and the Ricatti equation. Now consider the distance spheres

$$S\left(p,r\right)\doteqdot\left\{x\in M^n : d\left(x,p\right)=r\right\}.$$

Let h denote the second fundamental form of $S\left(p,r\right)$ as defined in (1.53). We have

$$\begin{aligned} h_{ij} &\doteqdot h\left(\frac{\partial}{\partial\theta^i},\frac{\partial}{\partial\theta^j}\right)=\left\langle\nabla_{\frac{\partial}{\partial\theta^i}}\frac{\partial}{\partial r},\frac{\partial}{\partial\theta^j}\right\rangle \\ &= -\left\langle\frac{\partial}{\partial r},\nabla_{\frac{\partial}{\partial\theta^i}}\frac{\partial}{\partial\theta^j}\right\rangle=-\Gamma_{ij}^n=\frac{1}{2}\frac{\partial}{\partial r}g_{ij} \end{aligned} \tag{1.82}$$

since $\frac{\partial}{\partial r}$ is the unit normal to $S\left(p,r\right)$ and $g_{in}=g_{jn}=0$. The mean curvature H of $S\left(p,r\right)$ is

$$H=-g^{ij}\Gamma_{ij}^n=\frac{1}{2}g^{ij}\frac{\partial}{\partial r}g_{ij}=\frac{\partial}{\partial r}\log\sqrt{\det g_{ij}}=\frac{\partial}{\partial r}\log\mathrm{J}\,. \tag{1.83}$$

Note that there was no need to normalize J (such as to satisfy (1.81)) to obtain this formula since the normalization only changes J by a multiplicative constant along a geodesic emanating from p which does not affect $\frac{\partial}{\partial r}\log\mathrm{J}$.

Exercise 1.87. *One may think of the distance spheres $S\left(p,r\right)$ as evolving under the hypersurface flow $\frac{\partial x}{\partial r}=\nu$, where $\nu=\frac{\partial}{\partial r}$ is the unit outward normal. Show that more generally, if the hypersurfaces are evolving by $\frac{\partial x}{\partial r}=\beta\nu$ for some function β, then $\frac{\partial}{\partial r}d\sigma=\beta H d\sigma$, where $d\sigma=\sqrt{\det g_{ij}}d\theta^1\wedge\cdots\wedge d\theta^{n-1}$ is the volume element of the hypersurface ((1.83) corresponds to the case $\beta=1$). Show also that*

$$\frac{\partial}{\partial r}g_{ij}=2\beta h_{ij}, \tag{1.84}$$

which generalizes (1.82).

EXERCISE 1.88. *Show that for r small enough:*

$$h_{ij} = \frac{1}{r} g_{ij} + O(r) \tag{1.85}$$

$$H = \frac{n-1}{r} + O(r). \tag{1.86}$$

In polar coordinates, the Laplacian is

$$\begin{aligned} \Delta &= g^{ab}\left(\frac{\partial^2}{\partial x^a \partial x^b} - \Gamma_{ab}^c \frac{\partial}{\partial x^c}\right) \\ &= \frac{\partial^2}{\partial r^2} + H\frac{\partial}{\partial r} + \Delta_{S(p,r)} = \frac{\partial^2}{\partial r^2} + \frac{\partial}{\partial r}\log\sqrt{\det g}\frac{\partial}{\partial r} + \Delta_{S(p,r)} \end{aligned} \tag{1.87}$$

since $\Gamma_{nn}^a = 0$ for $a = 1, \ldots, n$, and where $\Delta_{S(p,r)}$ is the Laplacian with respect to the induced metric on $S(p,r)$.

EXERCISE 1.89. *Show that one can also derive (1.87) directly from (1.57).*

We compute

$$\begin{aligned} \frac{\partial}{\partial r} h_{ij} &= -\left\langle \frac{\partial}{\partial r}, \nabla_{\frac{\partial}{\partial r}} \nabla_{\frac{\partial}{\partial \theta^i}} \frac{\partial}{\partial \theta^j} \right\rangle \\ &= -\left\langle \operatorname{Rm}\left(\frac{\partial}{\partial r}, \frac{\partial}{\partial \theta^i}\right)\frac{\partial}{\partial \theta^j}, \frac{\partial}{\partial r}\right\rangle - \left\langle \frac{\partial}{\partial r}, \nabla_{\frac{\partial}{\partial \theta^i}} \nabla_{\frac{\partial}{\partial r}} \frac{\partial}{\partial \theta^j}\right\rangle. \end{aligned}$$

Since

$$\begin{aligned} \left\langle \frac{\partial}{\partial r}, \nabla_{\frac{\partial}{\partial \theta^i}} \nabla_{\frac{\partial}{\partial r}} \frac{\partial}{\partial \theta^j}\right\rangle &= \frac{\partial}{\partial \theta^i}\left\langle \frac{\partial}{\partial r}, \nabla_{\frac{\partial}{\partial r}} \frac{\partial}{\partial \theta^j}\right\rangle - \left\langle \nabla_{\frac{\partial}{\partial \theta^i}} \frac{\partial}{\partial r}, \nabla_{\frac{\partial}{\partial r}} \frac{\partial}{\partial \theta^j}\right\rangle \\ &= -h_{ik} g^{k\ell} h_{\ell j} \end{aligned}$$

(for $\left\langle \frac{\partial}{\partial r}, \nabla_{\frac{\partial}{\partial r}} \frac{\partial}{\partial \theta^j}\right\rangle = 0$ and $\nabla_{\frac{\partial}{\partial \theta^i}} \frac{\partial}{\partial r} = \nabla_{\frac{\partial}{\partial r}} \frac{\partial}{\partial \theta^i}$), we obtain the **Ricatti equation**

$$\boxed{\frac{\partial}{\partial r} h_{ij} = -R_{nijn} + h_{ik} g^{k\ell} h_{\ell j}} \tag{1.88}$$

where $R_{nijn} \doteqdot \left\langle \operatorname{Rm}\left(\frac{\partial}{\partial r}, \frac{\partial}{\partial \theta^i}\right)\frac{\partial}{\partial \theta^j}, \frac{\partial}{\partial r}\right\rangle$. Since $\frac{\partial}{\partial r} H = g^{ij} \frac{\partial}{\partial r} h_{ij} - \frac{\partial}{\partial r} g_{ij} \cdot h_{ij}$ and $\frac{\partial}{\partial r} g_{ij} = 2h_{ij}$, tracing this equation yields

$$\boxed{\frac{\partial}{\partial r} H = -\operatorname{Rc}\left(\frac{\partial}{\partial r}, \frac{\partial}{\partial r}\right) - |h|^2.} \tag{1.89}$$

EXERCISE 1.90 (Evolution of mean curvature for a hypersurface flow). *The above formula is a special case (where $\beta = 1$) of the fact that under the hypersurface flow $\frac{\partial x}{\partial r} = \beta \nu$, we have the equation*

$$\frac{\partial}{\partial r} H = -\Delta \beta - |h|^2 \beta - \operatorname{Rc}(\nu, \nu)\beta$$

where the Laplacian is with respect to the induced metric on the hypersurface. Prove this. Note that when $\beta = -H$ (the mean curvature flow), we have a heat-type equation for H:

$$\frac{\partial}{\partial r} H = \Delta H + |h|^2 H + \operatorname{Rc}(\nu, \nu) H.$$

In particular, if $\operatorname{Rc} \geq (n-1) K g$, then since $|h|^2 \geq \frac{1}{n-1} H^2$ we have

$$\boxed{\frac{\partial}{\partial r} \frac{H}{n-1} \leq K - \left(\frac{H}{n-1}\right)^2.} \tag{1.90}$$

Note that by Exercise 1.88, we have

$$\lim_{r \to 0_+} \frac{rH}{n-1} = 1.$$

Remark 1.91. *Since*

$$\left(\nabla_{\frac{\partial}{\partial r}} h\right)_{ij} = \frac{\partial}{\partial r} h_{ij} - \Gamma_{ni}^k h_{kj} - \Gamma_{nj}^k h_{ik},$$

and $\Gamma_{ni}^k = -h_i^k$, we deduce from (1.88) that

$$\boxed{\left(\nabla_{\frac{\partial}{\partial r}} h\right)_{ij} = -R_{nijn} - h_{ik} g^{k\ell} h_{\ell j}.}$$

Invariantly, we write this as

$$\left(\nabla_{\frac{\partial}{\partial r}} h\right)(X, Y) = -\left\langle \operatorname{Rm}\left(\frac{\partial}{\partial r}, X\right) Y, \frac{\partial}{\partial r}\right\rangle - h^2(X, Y) \tag{1.91}$$

for $X, Y \in TS(p, r)$.

4.4. Comparison with space forms and Bishop volume comparison theorem. It is useful to compare with the space forms (M_K^n, g_K). In this case the metric is given by

$$g_K = dr^2 + s_K(r)^2 g_{S^{n-1}} \tag{1.92}$$

where

$$s_K(r) \doteqdot \begin{cases} \frac{1}{\sqrt{K}} \sin\left(\sqrt{K} r\right) & \text{if } K > 0 \\ r & \text{if } K = 0 \\ \frac{1}{\sqrt{|K|}} \sinh\left(\sqrt{|K|} r\right) & \text{if } K < 0 \end{cases}. \tag{1.93}$$

Recall that more generally, if

$$g = dr^2 + \phi(r)^2 g_{S^{n-1}}$$

for some function ϕ, which is a rotationally symmetric metric, then the sectional curvatures are

$$K_{\text{rad}} = -\frac{\phi''}{\phi}, \quad \text{and } K_{\text{sph}} = \frac{1 - (\phi')^2}{\phi^2} \tag{1.94}$$

where K_{rad} or K_{sph} is the sectional curvature of planes containing (rad for radial) or perpendicular (sph for spherical) to the radial vector, respectively. Moreover, if $\phi : [0, \rho) \to [0, \infty)$, with $\phi(r) > 0$ for $r > 0$, then the metric g extends smoothly over the origin if and only if $\phi(0) = 0$ and $\phi'(0) = 1$. We easily check that for any $K \in \mathbb{R}$, if we take $\phi(r) = s_K(r)$, then $g = g_K$ is a constant sectional curvature K metric. If $K \leq 0$, then g_K is defined on $\mathbb{R}^n$; and if $K > 0$, then the metric g_K defined on $B\left(0, \pi/\sqrt{K}\right)$ extends smoothly to a metric on S^n by taking the 1-point compactification.

EXERCISE 1.92 (Curvatures of a rotationally symmetric metric). *Use moving frames and the Cartan structure equations to derive (1.94) for* $g = dr^2 + \phi(r)^2 g_{S^{n-1}}$.

HINT: Let $\{\eta^i\}_{i=1}^{n-1}$ be a local orthonormal coframe field for $\left(S^{n-1}, g_{S^{n-1}}\right)$ and let η_i^j be the corresponding connection 1-forms which satisfy

$$d\eta^i = \eta^j \wedge \eta_j^i$$
$$d\eta_j^i - \eta_i^k \wedge \eta_k^j = \mathrm{Rm}\left(g_{S^{n-1}}\right)_i^j = \eta^i \wedge \eta^j.$$

Use the orthonormal frame for g defined by $\omega^n = dr$ and $\omega^i = \phi(r)\,\eta^i$ for $i = 1, \ldots, n-1$. For the solution of this exercise, see §16.

Note that one way of deriving (1.94) is to consider the distance spheres. From (1.82) we have $h_{ij} = \frac{\phi'}{\phi} g_{ij}$. That is, the spheres are **totally umbillic** with principal curvatures κ equal to $\kappa = \frac{\phi'}{\phi}$. The intrinsic curvature of the hypersurface $S(p, r)$ is

$$K_{\text{intrinsic}} = \frac{1}{\phi^2}.$$

From the Gauss equations, we have $K_{\text{sph}} = K_{\text{intrinsic}} - \kappa^2$. Note that the mean curvature H of $S(p, r)$ is $H = (n-1)\frac{\phi'}{\phi}$. In particular for the constant curvature K metric given by (1.92), the mean curvature $H_K(r)$ of the distance sphere $S_K(p, r)$ is

$$H_K(r) \doteqdot \begin{cases} (n-1)\sqrt{K}\cot\left(\sqrt{K}r\right) & \text{if } K > 0 \\ \frac{n-1}{r} & \text{if } K = 0 \\ (n-1)\sqrt{|K|}\coth\left(\sqrt{|K|}r\right) & \text{if } K < 0 \end{cases}.$$

Since $H_K(r)$ is a solution to the equality case of (1.90), that is,

$$\frac{\partial}{\partial r}\frac{H_K}{n-1} = K - \left(\frac{H_K}{n-1}\right)^2, \tag{1.95}$$

and $\lim_{r\to 0_+} \frac{rH_K}{n-1} = 1$. An easy calculus exercise shows: $H_K = \frac{n-1}{r} + O(r)$; in fact, (1.86) is more general.

By the ODE comparison theorem, we have:

LEMMA 1.93 (Mean curvature of distance spheres comparison). *If the Ricci curvature of* (M^n, g) *satisfies the lower bound* $\mathrm{Rc} \geq (n-1) Kg$ *for some* $K \in \mathbb{R}$*, then the mean curvatures of the distance spheres* $S(p,r)$ *satisfy*

$$\boxed{H(r,\theta) \leq H_K(r).} \tag{1.96}$$

PROOF. To see (1.96) more clearly we compute from (1.90) and (1.95) that

$$\frac{\partial}{\partial r}(H - H_K) \leq -\frac{(H_K + H)}{n-1}(H - H_K). \tag{1.97}$$

Note from (1.86) that $(H - H_K)(r) = O(r)$. Integrating (1.97), we get that for any $r \geq \varepsilon > 0$,

$$(H - H_K)(r) \leq (H - H_K)(\varepsilon) \cdot \exp\left\{-\int_\varepsilon^r \frac{(H_K + H)}{n-1}(s)\, ds\right\}. \tag{1.98}$$

Clearly for all $r > 0$ we have

$$\lim_{\varepsilon \to 0}\left((H - H_K)(\varepsilon) \cdot \exp\left\{-\int_\varepsilon^r \frac{(H_K + H)}{n-1}(s)\, ds\right\}\right) = 0.$$

Hence (1.98) implies $(H - H_K)(r) \leq 0$ for all $r > 0$. □

Given a point $p_K \in M_K^n$, let $\psi_{p_K} : T_{p_K} M_K^n - \left\{\vec{0}\right\} \to S_{p_K}^{n-1}$ be the standard projection $\psi_{p_K}(v) \doteqdot \frac{v}{|v|}$. The volume element of the space form satisfies

$$d\mu_K \doteqdot \sqrt{\det (g_K)_{ij}}\, d\theta^1 \wedge \cdots \wedge d\theta^{n-1} \wedge dr = s_K(r)^{n-1}\, d\sigma_K \wedge dr,$$

where $d\sigma_K$ is the pull back by $\psi_{p_K} \circ \exp_{p_K}^{-1}$ of the standard volume form on the unit sphere $S_{p_K}^{n-1}$. If $\left\{\theta^i\right\}_{i=1}^{n-1}$ are coordinates on $S_{p_K}^{n-1}$ with $d\theta^1 \wedge \cdots \wedge d\theta^{n-1} = d\sigma_K$, then

$$\sqrt{\det (g_K)_{ij}} = s_K(r)^{n-1}.$$

When $K \leq 0$, the above formula holds for all $r > 0$ and when $K > 0$ we need to assume $r \in \left(0, \pi/\sqrt{K}\right)$.

Now consider a Riemannian manifold (M^n, g) with $\mathrm{Rc} \geq (n-1) Kg$. From (1.83) and (1.96) we obtain

$$\boxed{\frac{\partial}{\partial r} \log \frac{\sqrt{\det g_{ij}}}{s_K(r)^{n-1}} \leq 0.} \tag{1.99}$$

Assume the coordinates $\left\{\theta^i\right\}_{i=1}^{n-1}$ on S_p^{n-1} satisfy $\lim_{r \to 0_+} \frac{1}{r} \frac{\partial}{\partial \theta^i} \doteqdot e_i \in T_p M^n$ are orthonormal. Then we have

$$\lim_{r \to 0_+} \frac{\sqrt{\det g_{ij}}}{s_K(r)^{n-1}} = 1,$$

from which we conclude

$$\mathrm{J} = \sqrt{\det g_{ij}} \leq s_K(r)^{n-1}. \tag{1.100}$$

Without making any normalizing assumption on the coordinates $\{\theta^i\}$ this says

$$\mathrm{J}(\theta, r)\, d\Theta(\theta) \leq s_K(r)^{n-1}\, d\sigma_{S^{n-1}}(\theta).$$

In other words, if $d\mu$ and $d\mu_K$ (for the purpose of this discussion) denote the pull backs of the volume forms of g and g_K to T_pM^n and $T_{p_K}M_K^n$, which we identify with each other via choices of orthonormal frames, then independent of these frames we have

$$\boxed{d\mu \leq d\mu_K.}$$

Integrating this proves the Bishop Volume Comparison Theorem 1.62, at least within the cut locus. To see this result holds on the whole manifold, we argue as follows (see Peter Li's book [**340**]). Let

$$C(r) \doteqdot \left\{V \in T_pM : |V| = 1 \text{ and } \gamma_V(s) = \exp_p(sV),\ s \in [0, r],\ \text{is minimizing}\right\}.$$

It is easy to see that if $r_1 \leq r_2$, then $C(r_2) \subset C(r_1)$. Since the cut locus of p has measure zero and $\exp_p^*(d\mu) = \mathrm{J}\, d\Theta \wedge dr$ inside the cut locus of p, for any integrable function φ on a geodesic ball $B(p, \bar{r})$ we have

$$\int_{B(p,\bar{r})} \varphi(x)\, d\mu(x) = \int_0^{\bar{r}} \left(\int_{C(r)} \varphi\left(\exp_p(\theta, r)\right) \mathrm{J}(\theta, r)\, d\Theta(\theta) \right) dr$$

where $d\Theta$ is defined by (1.79). In particular, by (1.100)

$$\begin{aligned}
\mathrm{Vol}(B(p, \bar{r})) &= \int_0^{\bar{r}} \left(\int_{C(r)} \mathrm{J}(\theta, r)\, d\Theta(\theta) \right) dr \\
&\leq \int_0^{\bar{r}} \left(\int_{C(r)} s_K(r)^{n-1}\, d\sigma_{S^{n-1}}(\theta) \right) dr \\
&\leq \int_0^{\bar{r}} \left(\int_{S^{n-1}} s_K(r)^{n-1}\, d\sigma_{S^{n-1}}(\theta) \right) dr = \mathrm{Vol}_K(B(p_K, \bar{r})).
\end{aligned}$$

This completes the proof of (1.71).

EXERCISE 1.94. *Complete the proof of Theorem 1.62 by proving that*

$$\frac{\mathrm{Vol}(B(p, r))}{\mathrm{Vol}_K(B(p_K, r))}$$

is a nonincreasing function of r.

We also leave it as an exercise that the Bishop-Gromov relative volume comparison theorem may also be proved along these lines; see Corollary 1.66 for the statement.

4.5. Mean value inequality, Laplacian and Hessian comparison theorems. Now we are ready to give the proof of the Mean Value Inequality.

Proof of Proposition 1.75. Let (M^n, g) be a complete Riemannian manifold with $\mathrm{Rc} \geq 0$ and $f \leq 0$ be a Lipschitz function with $\Delta f \geq 0$ in the sense of distributions. (By adding a constant to f if necessary, we obtain the general case where f is bounded from above.) By the divergence theorem, we have

$$0 \leq \frac{1}{r^{n-1}} \int_{B(x,r)} \Delta f \, d\mu = \int_{\partial B(x,r)} \frac{\partial f}{\partial r} \frac{\sqrt{\det g_{ij}}}{r^{n-1}} d\theta,$$

where $d\theta \doteqdot d\theta^1 \wedge \cdots \wedge d\theta^{n-1}$. Since $\frac{\partial}{\partial r} \frac{\sqrt{\det g_{ij}}}{r^{n-1}} \leq 0$ and $f \leq 0$, we have

$$\begin{aligned} 0 &\leq \int_{\partial B(x,r)} \left(\frac{\partial f}{\partial r} \frac{\sqrt{\det g_{ij}}}{r^{n-1}} + f \frac{\partial}{\partial r} \frac{\sqrt{\det g_{ij}}}{r^{n-1}} \right) d\theta \\ &= \frac{d}{dr} \left(\frac{1}{r^{n-1}} \int_{\partial B(x,r)} f d\sigma \right) \end{aligned}$$

since $d\sigma = \sqrt{\det g_{ij}} d\theta$. Since $\lim_{r \to 0} \frac{1}{r^{n-1}} \int_{\partial B(x,r)} f d\sigma = n\omega_n f(x)$, where $n\omega_n$ is the volume of the unit $(n-1)$-sphere, integrating the above inequality over $[0, s]$ yields

$$s^{n-1} f(x) \leq \frac{1}{n\omega_n} \int_{\partial B(x,s)} f d\sigma.$$

Integrating this again, now over $[0, r]$, implies

$$f(x) \leq \frac{n}{n\omega_n r^n} \int_{B(x,r)} f d\mu.$$

□

Exercise 1.95. *Prove Proposition 1.76.*

Remark 1.96 (Laplacian of a rotationally symmetric metric). *From (1.87) the Laplacian of the metric $g = dr^2 + \phi(r)^2 g_{S^{n-1}}$ is*

$$\Delta = \frac{\partial^2}{\partial r^2} + (n-1) \frac{\phi'}{\phi} \frac{\partial}{\partial r} + \Delta_{S(p,r)}. \tag{1.101}$$

In general, if $r(x) \doteqdot d(x, p)$ is the distance function to p, then since r is constant on each sphere $\Delta_{S(p,r)}$, then from (1.87) we have the *Laplacian of the distance function is the radial derivative of the logarithm of the Jacobian (and is the mean curvature of the distance spheres)*

$$\boxed{\Delta r = H = \frac{\partial}{\partial r} \log \mathrm{J}\,.} \tag{1.102}$$

Hence, if $\mathrm{Rc} \geq (n-1) K g$, then

$$\Delta r \leq H_K. \tag{1.103}$$

This proves the Laplacian Comparison Theorem 1.59, again, assuming we are within the cut locus. To prove that (1.103) holds in the sense of distributions on all of M^n, we argue as follows (see [**340**], p. 26). For any nonnegative $\varphi \in C^\infty(M^n)$ with compact support

$$\int_{M^n} \varphi(x) H_K(d_p(x))\, d\mu(x) = \int_0^\infty \int_{C(r)} \varphi\left(\exp_p(\theta, r)\right) H_K(r) \, \mathrm{J}(\theta, r)\, d\Theta(\theta)\, dr.$$

Given a unit vector $\theta \in T_p M^n$, let r_θ be the largest value of r such that $s \mapsto \gamma_\theta(s) = \exp_p(s\theta)$ minimizes up to $s = r$. By the Fubini Theorem, we have

$$\int_{M^n} \varphi(x) H_K(d_p(x))\, d\mu(x) = \int_{S^{n-1}} \int_0^{r_\theta} \varphi\left(\exp_p(\theta, r)\right) H_K(r) \, \mathrm{J}(\theta, r)\, dr d\Theta(\theta).$$

Now for $0 < r < r_\theta$, by (1.96) and (1.102)

$$H_K(r) \, \mathrm{J}(\theta, r) \geq H(r, \theta) \, \mathrm{J}(\theta, r) = \frac{\partial}{\partial r} \mathrm{J}(\theta, r).$$

Hence

$$\begin{aligned}
\int_{M^n} \varphi(x) H_K(d_p(x))\, d\mu(x) &\geq \int_{S^{n-1}} \int_0^{r_\theta} \varphi\left(\exp_p(\theta, r)\right) \frac{\partial}{\partial r} \mathrm{J}(\theta, r)\, dr d\Theta(\theta) \\
&= -\int_{S^{n-1}} \int_0^{r_\theta} \frac{\partial}{\partial r} \left(\varphi \circ \exp_p\right)(\theta, r) \, \mathrm{J}(\theta, r)\, dr d\Theta(\theta) \\
&\quad + \int_{S^{n-1}} \varphi\left(\exp_p(\theta, r_\theta)\right) \mathrm{J}(\theta, r_\theta)\, d\Theta(\theta)
\end{aligned}$$

where we integrated by parts and used $\lim_{r\to 0} \mathrm{J}(\theta, r) = 0$. Since the last line is nonnegative, we have by the Gauss Lemma (1.75)

$$\int_{M^n} \varphi(x) H_K(d_p(x))\, d\mu(x) \geq -\int_{M^n} \langle \nabla\varphi, \nabla r\rangle\, d\mu = \int_{M^n} r\Delta\varphi d\mu$$

where the last equality follows from the fact that r is Lipschitz on M^n. This completes the proof of Theorem 1.59.

We take this opportunity to make some further remarks about the volume comparison theorem. If $\mathrm{Rc}(g) \geq 0$, then by (1.96) we have $H(r, \theta) \leq \frac{n-1}{r}$. Hence the area $A(r)$ of the distance sphere $S(p, r)$ satisfies

$$\frac{d}{dr} A(r) = \int_{S(r)} H d\sigma \leq \frac{n-1}{r} A(r).$$

Integrating this we see that for $s \geq r$

$$A(s) \leq A(r) \frac{s^{n-1}}{r^{n-1}}.$$

Combining this with

$$\mathrm{Vol}\, B(p, r) = \int_0^r A(\rho)\, d\rho \geq \int_0^r A(r) \frac{\rho^{n-1}}{r^{n-1}} d\rho = \frac{r}{n} A(r), \tag{1.104}$$

we obtain, on a Riemannian manifold with nonnegative Ricci curvature, for $s \geq r$

$$A(s) \leq n \frac{\operatorname{Vol} B(p,r)}{r^n} s^{n-1}. \tag{1.105}$$

Let

$$\operatorname{AVR}(g) \doteqdot \lim_{r \to \infty} \frac{\operatorname{Vol}(B(p,r))}{\omega_n r^n} \tag{1.106}$$

be the **asymptotic volume ratio**.

EXERCISE 1.97. *Show that for $s \leq r$*

$$A(s) \geq n \frac{\operatorname{Vol} B(p,r)}{r^n} s^{n-1} \geq n\omega_n \operatorname{AVR}(g) s^{n-1}.$$

Now we consider the Hessian in polar coordinates. We have

$$\begin{aligned} \nabla_n \nabla_n &= \frac{\partial^2}{\partial r^2} - \Gamma_{nn}^a \frac{\partial}{\partial x^a} = \frac{\partial^2}{\partial r^2} \\ \nabla_n \nabla_i &= \frac{\partial^2}{\partial r \partial \theta^i} - \Gamma_{ni}^a \frac{\partial}{\partial x^a} = \frac{\partial^2}{\partial r \partial \theta^i} + h_i^j \frac{\partial}{\partial \theta^j} \\ \nabla_i \nabla_j &= \frac{\partial^2}{\partial \theta^i \partial \theta^j} - \Gamma_{ij}^a \frac{\partial}{\partial x^a} = \nabla_i^S \nabla_j^S + h_{ij} \frac{\partial}{\partial r} \end{aligned}$$

where ∇^S is the intrinsic covariant derivative of the hypersurface $S(p,r)$. In particular, if $f = f(r)$ is a radial function, then

$$\begin{aligned} \nabla_n \nabla_n f &= \frac{\partial^2 f}{\partial r^2} \\ \nabla_n \nabla_i f &= 0 \\ \nabla_i \nabla_j f &= h_{ij} \frac{\partial f}{\partial r}. \end{aligned}$$

Note that the *Hessian of the distance function is the second fundamental form of the distance sphere, which in turn is the radial derivative of the metric, telling us the about the inner products of the Jacobi fields* $\frac{\partial}{\partial \theta^i}$:

$$\nabla_i \nabla_j r = h_{ij} = \frac{1}{2} \frac{\partial}{\partial r} g_{ij} = \frac{1}{2} \frac{\partial}{\partial r} \left\langle \frac{\partial}{\partial \theta^i}, \frac{\partial}{\partial \theta^j} \right\rangle_g . \tag{1.107}$$

More invariantly, if J_1 and J_2 are Jacobi fields along a geodesic $\gamma : [0, L] \to M^n$ without conjugate points and if $J_i(0) = \vec{0}$ and $\langle \nabla_{\dot{\gamma}} J_i(0), \dot{\gamma}(0) \rangle = 0$ for $i = 1, 2$, then we have

$$\boxed{\frac{1}{2} \frac{\partial}{\partial r} \langle J_1, J_2 \rangle = \nabla_{J_1} \nabla_{J_2} r = h(J_1, J_2).} \tag{1.108}$$

Note that in a manifold with constant sectional curvature K, if J is a Jacobi field along a unit speed geodesic γ with $J(0) = \vec{0}$ and $\langle \nabla_{\dot{\gamma}} J, \dot{\gamma} \rangle (0) = 0$, then

$$|J|(\gamma(r)) = |\nabla_{\dot{\gamma}} J|(0)\, s_K(r)$$

where $s_K(r)$ is defined in (1.93). In general we have the expansion (see [**103**], p.16)

$$|J|^2(\gamma(r)) = r^2 - \frac{1}{3}\langle \mathrm{Rm}(\nabla_{\dot\gamma}J, \dot\gamma)\dot\gamma, \nabla_{\dot\gamma}J\rangle(0)\cdot r^4 + O(r^5).$$

Now suppose $\operatorname{sect}(g) \geq K$. From (1.91) we have

$$\nabla_{\frac{\partial}{\partial r}} h \leq -Kg - h^2.$$

Since along a geodesic ray $\gamma : [0, L) \to M^n$ emanating from p we have

$$\lim_{r\to 0_+} rh\left(X_{\gamma(r)}, Y_{\gamma(r)}\right) = g(X_p, Y_p),$$

for parallel vector fields X and Y along γ, we obtain

$$\boxed{h(r,\theta) \leq h_K(r)\, g(r,\theta)} \tag{1.109}$$

where

$$h_K(r) \doteqdot \begin{cases} \sqrt{K}\cot\left(\sqrt{K}r\right) & \text{if } K>0 \\ \frac{1}{r} & \text{if } K=0 \\ \sqrt{|K|}\coth\left(\sqrt{|K|}r\right) & \text{if } K<0 \end{cases} \qquad = \frac{1}{n-1}H_K(r).$$

Hence if $f(r) = r$ is the distance function, then we get

$$\boxed{\nabla_i\nabla_j r = h_{ij} \leq h_K(r)\, g_{ij}} \tag{1.110}$$

for $i, j = 1, \ldots, n-1$ inside the cut locus when $\operatorname{sect}(g) \geq K$. This is a special case of the Hessian Comparison Theorem 1.82. We also obtain:

COROLLARY 1.98. *Let (M^n, g) be Riemannian manifold with* $\operatorname{sect}(g) \geq K$ *and $\gamma : [0, L] \to M^n$ be a unit speed geodesic. If J is a Jacobi field along γ with $J(0) = \vec{0}$ and $\langle \nabla_{\dot\gamma}J(0), \dot\gamma(0)\rangle = 0$, then*

$$|J(r)| \leq |\nabla_{\dot\gamma}J(0)| \cdot s_K(r).$$

PROOF. By our hypotheses, $\langle J(r), \dot\gamma(r)\rangle \equiv 0$ for all r. From (1.108) and (1.109) we have for all $r > 0$,

$$\frac{\partial}{\partial r}\left(\frac{|J(r)|}{s_K(r)}\right) = \left(h\left(\tfrac{J}{|J|}, \tfrac{J}{|J|}\right) - \frac{s_K'(r)}{s_K(r)}\right)\cdot\frac{|J|}{s_K(r)} \leq 0,$$

since $\frac{s_K'(r)}{s_K(r)} = h_K(r)$ (recall $s_K(r)$ is defined by (1.93).) The result now follows from $\lim_{r\to 0}\frac{|J(r)|}{s_K(r)} = |\nabla_{\dot\gamma}J(0)|$. □

More generally, applying standard ODE comparison theory to the Jacobi equation, one has the following.

THEOREM 1.99 (Rauch Comparison). *Let (M^n, g) and (M_0^n, g_0) be Riemannian manifolds and $\gamma : [0, L] \to M^n$ and $\gamma_0 : [0, L] \to M_0^n$ be unit speed geodesics. Suppose that γ_0 has no conjugate points and for any $r \in [0, L]$ and any $X \in T_{\gamma(r)}M^n$, $X_0 \in T_{\gamma(r)}M_0^n$ we have*

$$\operatorname{sect}\left(X \wedge \dot{\gamma}(r)\right) \leq \operatorname{sect}\left(X_0 \wedge \dot{\gamma}_0(r)\right).$$

If J and J_0 are Jacobi fields along γ and γ_0 with $J(0)$ and $J_0(0)$ tangent to γ and γ_0, and

$$\begin{aligned} |J(0)| &= |J_0(0)|, \\ \langle \nabla_{\dot{\gamma}} J(0), \dot{\gamma}(0)\rangle &= \langle \nabla_{\dot{\gamma}_0} J_0(0), \dot{\gamma}_0(0)\rangle, \\ |\nabla_{\dot{\gamma}} J(0)| &= |\nabla_{\dot{\gamma}_0} J_0(0)|, \end{aligned}$$

then

$$|J(r)| \geq |J_0(r)|. \tag{1.111}$$

For the proof of this see Chapter 1 of [**103**].

EXERCISE 1.100. *Show that the Rauch Comparison Theorem may be used to prove the Hessian Comparison Theorem. Similarly, show that the Bishop Volume Comparison Theorem implies the Laplacian Comparison Theorem.*

EXERCISE 1.101. *Show that if* $\operatorname{sect}(g) \leq K$, *then* $\nabla_i \nabla_j r \geq h_K(r)\, g_{ij}$.

EXERCISE 1.102. *Under the assumption of* $\operatorname{sect}(g) \geq K$ *prove the following Hessian comparisons:*

$$\nabla\nabla \cos\left(\sqrt{K} r\right) \geq -K \cos\left(\sqrt{K} r\right) g \text{ if } K > 0 \tag{1.112}$$

$$\nabla\nabla \left(r^2\right) \leq 2g \text{ if } K = 0 \tag{1.113}$$

$$\nabla\nabla \cosh\left(\sqrt{|K|} r\right) \leq |K| \cosh\left(\sqrt{|K|} r\right) g \text{ if } K < 0. \tag{1.114}$$

Here the comparisons are the same in any direction, not just the spherical directions as in the case of the distance function. Note that

$$r^2 = \lim_{r \to 0_+} \frac{2}{K}\left(1 - \cos\left(\sqrt{K} r\right)\right) = \lim_{r \to 0_-} \frac{2}{|K|}\left(\cosh\left(\sqrt{|K|} r\right) - 1\right).$$

The reason the sign in the equality is reversed when $K > 0$ is because $\frac{d}{dx}\cos x = -\sin x < 0$.

5. First and second variation of arc length and energy formulas

5.1. Arc length. We recall the first and second variation of arc length formulas. Let $\gamma_r : [a, b] \to M^n$, $r \in \mathcal{J} \subset \mathbb{R}$, be a 1-parameter family of paths. From this we may define the map $\Gamma : [a, b] \times \mathcal{J} \to M^n$ by

$$\Gamma(s, r) \doteqdot \gamma_r(s).$$

We define the vector fields R and S along γ_r by $R \doteqdot \Gamma_* (\partial/\partial r)$ and $S \doteqdot \Gamma_* (\partial/\partial s)$. We call R the **variation vector field** and S the **tangent vector field**. The length of γ_r is given by

$$\mathrm{L}(\gamma_r) \doteqdot \int_a^b |S(\gamma_r(s))| \, ds.$$

The **first variation of arc length formula** is given by

LEMMA 1.103 (1st variation of arc length). *Suppose* $0 \in \mathcal{J}$. *If* γ_0 *is parametrized by arc length, that is,* $|S(\gamma_0(s))| \equiv 1$, *then*

$$\boxed{\left.\frac{d}{dr}\right|_{r=0} \mathrm{L}(\gamma_r) = -\int_a^b \langle R, \nabla_S S \rangle \, ds + \langle R, S \rangle|_a^b .} \tag{1.115}$$

PROOF. We compute

$$\begin{aligned} \left.\frac{d}{dr}\right| \mathrm{L}(\gamma_r) &= \frac{1}{2} \int_a^b |S|^{-1} R \langle S, S \rangle \, ds = \int_a^b |S|^{-1} \langle S, \nabla_R S \rangle \, ds \\ &= \int_a^b \langle S/|S|, \nabla_S R \rangle \, ds \end{aligned} \tag{1.116}$$

since $\nabla_R S - \nabla_S R = [R, S] = \Gamma_* ([\partial/\partial r, \partial/\partial s]) = 0$. Using $|S| \equiv 1$ and integrating by parts yields the desired formula. □

COROLLARY 1.104. *If* $\gamma_r : [0, b] \to M^n$, $r \in \mathcal{J} \subset \mathbb{R}$, *is a 1-parameter family of paths emanating from a fixed point* $p \in M^n$ *(i.e.,* $\gamma_r(0) = p$*) and* γ_0 *is a geodesic parametrized by arc length, then*

$$\left.\frac{d}{dr}\right|_{r=0} \mathrm{L}(\gamma_r) = \left\langle \left.\frac{\partial}{\partial r}\right|_{r=0} \gamma_r(b), \frac{\partial \gamma_0}{\partial s}(b) \right\rangle . \tag{1.117}$$

Observe that at a point where the distance function $r(x) \doteqdot d(x, p)$ is smooth (i.e., outside the cut locus), the first variation formula (1.117) implies

$$\boxed{\nabla r(x) = \frac{\partial \gamma_0}{\partial s}(b)} \tag{1.118}$$

where $\gamma_0 : [0, b] \to M^n$ is the unique unit speed minimal geodesic from p to x. This is because if γ_0 is a minimal geodesic, then $\mathrm{L}(\gamma_r) \geq d(p, \gamma_r(b))$ with $\mathrm{L}(\gamma_0) = d(p, \gamma_0(b))$. Hence

$$\left\langle \nabla r, \left.\frac{\partial}{\partial r}\right|_{r=0} \gamma_r(b) \right\rangle = \left.\frac{d}{dr}\right|_{r=0} \mathrm{L}(\gamma_r) = \left\langle \left.\frac{\partial}{\partial r}\right|_{r=0} \gamma_r(b), \frac{\partial \gamma_0}{\partial s}(b) \right\rangle .$$

This is an alternative way to view a formula we already knew from the Gauss Lemma.

REMARK 1.105. *If we do **not** assume* γ_0 *is parametrized by arc length, then we have*

$$\left.\frac{d}{dr}\right|_{r=0} \mathrm{L}(\gamma_r) = -\int_a^b \left\langle R, \nabla_S \left(\frac{S}{|S|} \right) \right\rangle ds + \left. \left\langle R, \frac{S}{|S|} \right\rangle \right|_a^b .$$

Hence, among all paths fixing two endpoints, the critical points of the length functional are the ***geodesics*** γ*, which satisfy*

$$\nabla_{\dot{\gamma}}\left(\frac{\dot{\gamma}}{|\dot{\gamma}|}\right)=0.$$

Now suppose we have a 2-parameter family of paths $\gamma_{q,r}:[a,b]\to M^n$, $q\in\mathcal{I}\subset\mathbb{R}$ and $r\in\mathcal{J}\subset\mathbb{R}$. Define $\Phi:[a,b]\times\mathcal{I}\times\mathcal{J}\to M^n$ by

$$\Phi(s,q,r)\doteqdot\gamma_{q,r}(s)$$

and $Q\doteqdot\Phi_*(\partial/\partial q)$, $R\doteqdot\Phi_*(\partial/\partial r)$ and $S\doteqdot\Phi_*(\partial/\partial s)$. The **second variation of arc length formula** is given by

LEMMA 1.106 (2nd variation of arc length). *Suppose* $0\in\mathcal{I}$ *and* $0\in\mathcal{J}$. *If* $\gamma_{0,0}$ *is parametrized by arc length, then*

$$\begin{aligned}(1.119)\quad &\left.\frac{\partial^2}{\partial q\partial r}\right|_{(q,r)=(0,0)}\mathrm{L}(\gamma_{q,r})\\ &=\int_a^b\left(\langle\nabla_SQ,\nabla_SR\rangle-\langle\nabla_SQ,S\rangle\langle\nabla_SR,S\rangle-\langle\mathrm{Rm}(Q,S)S,R\rangle\right)ds\\ &-\int_a^b\langle\nabla_QR,\nabla_SS\rangle\,ds+\langle\nabla_QR,S\rangle|_a^b\,.\end{aligned}$$

PROOF. Differentiating (1.116), we compute that

$$\begin{aligned}&\left.\frac{\partial^2}{\partial q\partial r}\right|_{(q,r)=(0,0)}\mathrm{L}(\gamma_{q,r})\\ &=\left.\frac{\partial}{\partial q}\right|_{(q,r)=(0,0)}\int_a^b\langle S/|S|,\nabla_SR\rangle\,ds=\int_a^bQ\,\langle S/|S|,\nabla_SR\rangle\,ds\\ &=\int_a^b\left(\langle S/|S|,\nabla_Q\nabla_SR\rangle+\langle\nabla_Q(S/|S|),\nabla_SR\rangle\right)ds\\ &=\int_a^b\left(\langle\mathrm{Rm}(Q,S)R,S/|S|\rangle+\langle S/|S|,\nabla_S\nabla_QR\rangle\right)ds\\ &+\int_a^b|S|^{-1}\langle\nabla_QS,\nabla_SR\rangle\,ds-\int_a^b|S|^{-3}\langle\nabla_QS,S\rangle\langle S,\nabla_SR\rangle\,ds\end{aligned}$$

and the result follows from an integration by parts. □

COROLLARY 1.107. *If* γ_r *is a 1-parameter family of paths with fixed end points and such that* γ_0 *is a geodesic parametrized by arc length, then*

$$(1.120)\qquad\left.\frac{\partial^2}{\partial r^2}\right|_{r=0}\mathrm{L}(\gamma_r)=\int_a^b\left(\left|(\nabla_SR)^\perp\right|^2-\langle\mathrm{Rm}(R,S)S,R\rangle\right)ds.$$

where $(\nabla_SR)^\perp$ *is the projection of* ∇_SR *onto* $S^\perp$*, i.e.,* $(\nabla_SR)^\perp\doteqdot\nabla_SR-\langle\nabla_SR,S\rangle S$.

In particular, we have

Corollary 1.108. *If in addition, (M^n, g) has nonpositive sectional curvature and the paths γ_r are smooth and closed, then $\frac{\partial^2}{\partial r^2}\Big|_{r=0} \mathrm{L}(\gamma_r) \geq 0$. That is, any smooth closed geodesic γ_0 is **stable**.*

With the idea of minimizing in a homotopy class of an element of $\pi_1(M^n)$ and some simple linear algebra, one also obtains:

Theorem 1.109 (Synge). *If (M^n, g) is an even-dimensional, orientable, closed Riemannian manifold with positive sectional curvature, then M^n is simply connected.*

Proof. Suppose Θ is a nontrivial free homotopy class of loops. Then since M^n is compact, there exists a smooth closed geodesic γ representing Θ with minimal length among all such loops (see p. 99 of [**103**] for the proof.) Fix a point $p \in \gamma$ and consider parallel translation around γ to obtain a linear isometry of an even-dimensional vector space: $\iota : T_pM \to T_pM$. Now $\iota(\dot{\gamma}) = \dot{\gamma}$ because γ is a geodesic. Since M^n is orientable, $\iota : \dot{\gamma}^\perp \to \dot{\gamma}^\perp$ is an orthogonal transformation with determinant 1 of an odd-dimensional vector space with an inner product. Now the eigenvalues of an orthogonal matrix are complex numbers with length 1 coming in conjugate pairs. Since the dimension of $\dot{\gamma}^\perp$ is odd and the determinant of ι is 1, we conclude that 1 is an eigenvalue. Hence there exists a smooth parallel unit vector field R along γ normal to $\dot{\gamma}$. The second variation formula (1.120) implies that

$$\frac{\partial^2}{\partial r^2}\bigg|_{r=0} \mathrm{L}(\gamma_r) = -\int_a^b \langle \mathrm{Rm}(R, S) S, R \rangle \, ds < 0$$

since the sectional curvatures are positive. This contradicts the fact that γ has minimal length in its free homotopy class. Hence M^n is simply connected. ☐

If $\gamma_q : [0, r] \to M^n$ is one-parameter family of paths, $q \in (-\varepsilon, \varepsilon)$, γ_0 is a unit speed geodesic, and $Q(0) = \vec{0}$, then

(1.121)
$$\frac{\partial^2}{\partial q^2}\bigg|_{q=0} \mathrm{L}(\gamma_q) - \langle \nabla_Q Q, S \rangle (r) = \int_0^r \left(\left| (\nabla_S Q)^\perp \right|^2 - \langle \mathrm{Rm}(Q, S) S, Q \rangle \right) ds.$$

In particular, given $V \in T_{\gamma_0(r)}M$, we extend V along γ_0 by defining

$$Q(\gamma_0(s)) \doteqdot \frac{s}{r} V(\gamma_0(s))$$

where $V(\gamma_0(s))$ is the parallel translation of V along γ. We then have

$$\frac{\partial^2}{\partial q^2}\bigg|_{q=0} \mathrm{L}(\gamma_q) - \langle \nabla_Q Q, S \rangle (r) = \frac{1}{r} \left| V^\perp \right|^2 - \int_0^r \langle \mathrm{Rm}(Q, S) S, Q \rangle \, ds \qquad (1.122)$$

where $V^{\perp} \doteqdot V - \langle V, S\rangle S$. Now let $\beta : (-\varepsilon, \varepsilon) \to M^n$ where $\beta(q) = \gamma_q(r)$ so that $\dot{\beta}(q) = Q$ and in particular $\dot{\beta}(0) = V$. We have

$$d(\beta(q), p) \leq \mathrm{L}(\gamma_q)$$
$$d(\beta(0), p) = \mathrm{L}(\gamma_0).$$

That, is the function $q \mapsto d(\beta(q), p)$ is a lower support function for the function $q \mapsto \mathrm{L}(\gamma_q)$ at $q = 0$.

DEFINITION 1.110. *Suppose that $u \in C^0(M^n)$ and $V \in T_pM$. Let $\beta_V : (-\varepsilon, \varepsilon) \to M^n$ be the constant speed geodesic with $\beta(0) = p$ and $\dot{\beta}(0) = V \in$. If $v : (-\varepsilon, \varepsilon) \to \mathbb{R}$ is a C^2 function such that $u(\beta(q)) \leq v(q)$ for $q \in (-\varepsilon, \varepsilon)$ and $u(\beta(0)) = v(0)$, then we say that*

$$\nabla_V \nabla_V u \leq v''(0)$$

***in the sense of support functions**.*

EXERCISE 1.111. *Show that if $u : M^n \to \mathbb{R}$ satisfies $\nabla_V \nabla_V u \leq 0$ for every $V \in TM$ in the sense of support functions, then u is concave; that is, for every unit speed geodesic $\beta : [a, b] \to M^n$ we have*

$$u(\beta((1-s)a + sb)) \geq (1-s)u(\beta(a)) + su(\beta(b))$$

for all $s \in [0, 1]$.

Now continuing our discussion above, assume β is a geodesic so that $\nabla_Q Q = 0$. In particular, if the sectional curvatures are nonnegative and $r(x) = d(x, p)$ is the distance function, then (compare (1.110))

$$\nabla_V \nabla_V r \leq \left.\frac{\partial^2}{\partial q^2}\right|_{q=0} \mathrm{L}(\gamma_q) \leq \frac{1}{r}\left|V^{\perp}\right|^2 \tag{1.123}$$

in the sense of support functions. Note that this inequality holds in the usual C^2 sense outside the cut locus of p.

EXERCISE 1.112. *Generalize (1.123) to the case* sect $\geq K$, *where $K \in \mathbb{R}$.*

EXERCISE 1.113. *Show that equation (1.123) implies (compare (1.113))*

$$\nabla_V \nabla_V \left(r^2\right) \leq 2|V|^2. \tag{1.124}$$

HINT*: assuming β is a geodesic, we have*

$$\left.\frac{\partial^2}{\partial q^2}\right|_{q=0} \mathrm{L}(\gamma_q)^2 \leq 2\left|V^{\perp}\right|^2 + 2\left(\left.\frac{\partial}{\partial q}\right|_{q=0} \mathrm{L}(\gamma_q)\right)^2. \tag{1.125}$$

If $\gamma : [a, b] \to M^n$ is a path and Q and R are vector fields along γ, we define the **index form** by

$$I(Q, R) \doteqdot \int_a^b \left(\langle \nabla_S Q, \nabla_S R\rangle - \langle \nabla_S Q, S\rangle \langle \nabla_S R, S\rangle - \langle \mathrm{Rm}(Q, S) S, R\rangle\right) ds.$$

Note that if either $\langle Q,S\rangle \equiv 0$ on γ or $\langle R,S\rangle \equiv 0$ on γ, then

$$I(Q,R) = \int_a^b \left(\langle \nabla_S Q, \nabla_S R\rangle - \langle \mathrm{Rm}(Q,S)\,S, R\rangle\right) ds.$$

If $\gamma_{q,r}$ is a 1-parameter family of paths with fixed endpoints and if $\gamma_{0,0}$ is a unit speed geodesic, then

$$\left.\frac{\partial^2}{\partial q \partial r}\right|_{(q,r)=(0,0)} \mathrm{L}(\gamma_{q,r}) = I(Q,R).$$

5.2. Energy. Analogous to the length functional is the **energy functional** of a path:

$$\mathrm{Energy}(\gamma) \doteqdot \int_a^b \left|\frac{d\gamma}{ds}(s)\right|^2 ds.$$

Given a 1-parameter family of paths $\gamma_r : [a,b] \to M^n$, $r \in \mathcal{J} \subset \mathbb{R}$, the first variation formula is given by (R and S defined the same as above)

(1.126)

$$\frac{1}{2}\left.\frac{d}{dr}\right|_{r=0} \mathrm{Energy}(\gamma_r) = \int_a^b \langle \nabla_S R, S\rangle\, ds = -\int_a^b \langle R, \nabla_S S\rangle\, ds + \langle R,S\rangle|_a^b\,.$$

Note we have not assumed that γ_0 is parameterized by arc length. Hence the critical points of the energy functional, among all paths fixing two endpoints, are the **constant speed geodesics** γ, which satisfy

$$\nabla_{\dot\gamma}\dot\gamma = 0.$$

The speed of γ is constant since $\dot\gamma\,|\dot\gamma|^2 = 2\,\langle \nabla_{\dot\gamma}\dot\gamma, \dot\gamma\rangle \equiv 0$.

Now let $\gamma_{q,r} : [a,b] \to M^n$, $q \in \mathcal{I} \subset \mathbb{R}$ and $r \in \mathcal{J} \subset \mathbb{R}$, be a 2-parameter family of paths. A similar computation as in the proof of Lemma 1.106 shows that the second variation is given by

$$\begin{aligned}\left.\frac{\partial^2}{\partial q \partial r}\right|_{(q,r)=(0,0)} \mathrm{Energy}(\gamma_{q,r}) &= \int_a^b Q\,\langle S, \nabla_S R\rangle\, ds \\ &= \int_a^b \left(\langle S, \nabla_Q \nabla_S R\rangle + \langle \nabla_Q S, \nabla_S R\rangle\right) ds \\ &= \int_a^b \left(\langle \nabla_S Q, \nabla_S R\rangle + \langle \mathrm{Rm}(Q,S)\,R, S\rangle\right) ds \\ &\quad - \int_a^b \langle \nabla_Q R, \nabla_S S\rangle\, ds + \langle \nabla_Q R, S\rangle|_a^b\end{aligned} \tag{1.127}$$

5.3. Jacobi fields. Recall that a Jacobi field J is a variation of geodesics and satisfies the Jacobi equation:

$$\boxed{\nabla_S \nabla_S J + R(J,S)\,S = 0}$$

Given a geodesic $\gamma : [0,L] \to M^n$ without conjugate points and vectors $A \in T_{\gamma(0)}M^n$ and $B \in T_{\gamma(L)}M^n$ with $\langle A,S\rangle = 0$ and $\langle B,S\rangle = 0$, there exists a unique Jacobi field J with $J(0) = A$ and $J(L) = B$.

Moreover,

LEMMA 1.114. *In the space $\Xi_{A,B}$ of vector fields X along γ with $X(0) = A$ and $X(L) = B$, the Jacobi field minimizes the index form:*

$$\mathcal{I}(X) \doteqdot I(X, X) = \int_0^L \left(|\nabla_S X|^2 - \langle \mathrm{Rm}(X, S) S, X \rangle \right) ds.$$

REMARK 1.115. *The idea is that geodesic variations should minimize the second variation among all variations with given endpoint values.*

PROOF. Note that if X and Y are vector fields along γ, then

$$\frac{1}{2} \frac{d}{dt}\bigg|_{t=0} \mathcal{I}(X + tY) = -\int_0^L \langle (\nabla_S \nabla_S X + \mathrm{Rm}(X, S) S), Y \rangle \, ds.$$

Hence the critical points of $\mathcal{I}$ on $\Xi_{A,B}$ are the Jacobi fields. We also see that if γ has no conjugate points, then

$$\frac{1}{2} \frac{d^2}{dt^2}\bigg|_{t=0} \mathcal{I}(X + tY) = \int_0^L \left(|\nabla_S Y|^2 - \langle \mathrm{Rm}(Y, S) S, Y \rangle \right) ds > 0 \tag{1.128}$$

when $Y(0) = 0$, $Y(L) = 0$ and $Y \not\equiv 0$, so that the index form $\mathcal{I}$ is convex. (Note that the tangent space $T_X \Xi_{A,B}$ is the space of all vector fields along γ which vanish at its endpoints.) Hence the Jacobi fields minimize $\mathcal{I}$ in $\Xi_{A,B}$ (see Lemma 1.169).

We now give a variational proof of inequality (1.128). Given a unit speed geodesic $\gamma : [0, L] \to M^n$ without conjugate points and a vector field Y along γ, define

$$\mathcal{I}_t(Y) \doteqdot \int_0^t \left(|\nabla_S Y|^2 - \langle \mathrm{Rm}(Y, S) S, Y \rangle \right) ds$$

for $t \in [0, L]$. Normalize the index by defining

$$\iota(t) \doteqdot \inf_{\substack{Z(0)=0,\ Z(t)=0 \\ Z \not\equiv 0}} \frac{\mathcal{I}_t(Z)}{\int_0^t |Z(s)|^2 \, ds}.$$

Note that $\lim_{t \to 0_+} \iota(t) = \infty$. This is because

$$\frac{d}{ds} |Z| \leq |\nabla_S Z|, \quad \langle \mathrm{Rm}(Z, S) S, Z \rangle \leq C |Z|^2$$

and

$$\lim_{t \to 0} \lambda_1([0, t]) = \lim_{t \to 0} \frac{\pi^2}{t^2} = \infty$$

where λ_1 is the first eigenvalue of d^2/ds^2 with Dirichlet boundary conditions.

For $t \in (0, L]$ where $\gamma|_{[0,t]}$ is minimizing (e.g., for $t > 0$ small enough), we have $\iota(t) \geq 0$ since $\mathcal{I}_t(Z)$ is a second variation of $\gamma|_{[0,t]}$ vanishing at the endpoints 0 and t. Now if for some $t_0 \in (0, L]$ we have $\iota(t_0) = 0$, then $\mathcal{I}_{t_0}(Z_0) = 0$ for some Z_0 with $Z_0(0) = 0$, $Z_0(t) = 0$, and $Z_0 \not\equiv 0$. By considering the Euler-Lagrange equation for $E(Z) \doteqdot \mathcal{I}_{t_0}(Z) / \int_0^{t_0} |Z(s)|^2 \, ds$ at Z_0, we have for all W vanishing at 0 and t_0

$$
\begin{aligned}
0 &= \frac{1}{2} \left. \frac{d}{du} \right|_{u=0} E(Z_0 + uW) \\
&= -\frac{1}{\int_0^t |Z_0(s)|^2 \, ds} \int_0^t \langle (\nabla_S \nabla_S Z_0 + \operatorname{Rm}(Z_0, S) S), W \rangle \, ds,
\end{aligned}
$$

since $\mathcal{I}_{t_0}(Z_0) = 0$. Thus Z_0 is a nontrivial Jacobi field along $\gamma|_{[0,t_0]}$ with $Z_0(0) = 0$ and $Z_0(t_0) = 0$. This contradicts the assumption that there are no conjugate points along γ. Hence $\iota(t) > 0$ for all $t \in (0, L]$. □

REMARK 1.116. *Since in the of $\iota(t)$ we make take Z to be piecewise smooth, we see that $\iota(t)$ is nonincreasing in t.*

We briefly return to the Laplacian and Hessian comparison theorems, now from the point of view of Jacobi fields. (See Exercise 2.16 in [**98**].) Let $p \in M^n$, γ be a unit speed geodesic emanating from p, and $r(x) \doteqdot d(x, p)$. Let J be a Jacobi field along γ perpendicular to $\dot{\gamma}$ and with $J(p) = 0$. Then for any s such that γ is minimizing past $\gamma(s)$, we have

$$
I\left(J|_{[0,s]}, J|_{[0,s]}\right) = \left(\nabla^2_{J(s),J(s)} r\right)(\gamma(s)) = h(J(s), J(s)) \tag{1.129}
$$

where h is the second fundamental form of the distance sphere $S(p, s)$. In particular, given any orthonormal basis $\{e_i\}_{i=1}^{n-1}$ for $\dot{\gamma}^{\perp}$ at $\gamma(s)$, let J_i denote the Jacobi fields along γ with $J_i(0) = 0$ and $J_i(s) = e_i$. We have

$$
\Delta r = \sum_{i=1}^{n-1} I(J_i, J_i).
$$

Since Jacobi fields minimize the index form, we have for $x \notin \operatorname{Cut}(p)$

$$
\Delta r(x) \leq \sum_{i=1}^{n-1} I(Y_i, Y_i)(x) \tag{1.130}
$$

for any vector fields Y_i along the minimal geodesic γ joining p to x with $Y_i(0) = 0$ and $\{Y_i(r(x))\}_{i=1}^{n-1}$ orthonormal.

We now prove (1.129). Let $\beta : [0, \varepsilon] \to M^n$ be a geodesic with $\beta(0) = \gamma(s)$ and $\dot{\beta}(0) = J(s)$. Since γ is minimizing and $\gamma(s) \notin \operatorname{Cut}(p)$, there exists a unique smooth 1-parameter family of geodesics

$$
\gamma_u : [0, s] \to M^n
$$

with $\gamma_u(0) = p$ and $\gamma_u(s) = \beta(u)$ for $u \in [0, \varepsilon]$. We have $\left. \frac{\partial}{\partial u} \right|_{u=0} \gamma_u(t) = J(t) \in T_{\gamma(t)} M^n$ for $t \in [0, s]$. Now we apply the second variation formula

(1.119) to obtain

$$\begin{aligned}\left(\nabla^2_{J(s),J(s)}r\right)(\gamma(s)) &= \left.\frac{\partial^2}{\partial u^2}\right|_{u=0} r(\beta(u)) = \left.\frac{\partial^2}{\partial u^2}\right|_{u=0} \mathrm{L}(\gamma_u)\\ &= \int_0^s \left(|\nabla_T J|^2 - \langle \mathrm{Rm}(J,T)T,J\rangle\right) dt = J|_{[0,s]}, J|_{[0,s]}\end{aligned}$$

where we used $\nabla_{\dot\gamma}\dot\gamma = 0$, $\langle J, \dot\gamma\rangle = 0$ and $\nabla_{\dot\beta}\dot\beta = 0$.

REMARK 1.117. *Perelman has defined a new space-time geometry for solutions of the Ricci flow (see Volume 2.) There we shall carry out analogous calculations for the first and second variation of the $\mathcal{L}$-length.*

6. Geometric applications of second variation

6.1. Toponogov comparison theorem. As a consequence of our discussion in §5, we have the following (see also [**308**] and pp. 105-106 of [**234**]).

LEMMA 1.118. *Let (M^n, g) be a complete Riemannian manifold with nonnegative sectional curvature and $p \in M^n$. If $\beta : (a,b) \to M^n$ is a unit speed geodesic, then the function*

$$\phi : (a,b) \to \mathbb{R}$$

defined by

$$\phi(r) \doteqdot r^2 - d(\beta(r), p)^2$$

is convex.

PROOF. This is essentially contained in the arguments leading up to Exercise 1.113; for completeness we repeat part of the arguments. Given $r_0 \in (a,b)$, let $\gamma_r : [0,L] \to M^n$ be a 1-parameter family of paths from p to $\beta(r)$ with $\gamma_{r_0} : [0,L] \to M^n$ a unit speed minimal geodesic from p to $\beta(r_0)$ and

$$\left.\frac{\partial}{\partial r}\right|_{r=r_0} \gamma_r(s) = \frac{s}{L} V(\gamma(s)),$$

where V is the parallel translation of $\dot\beta(r_0) \in T_{\gamma(L)}M$ along γ. By (1.125) we have

$$\left.\frac{d^2}{dr^2}\right|_{r=r_0} \left(r^2 - \mathrm{L}(\gamma_r)^2\right) \geq 0$$

since $|V|^2 = 1$. Since

$$r^2 - d(\beta(r), p)^2 \geq r^2 - \mathrm{L}(\gamma_r)^2, \quad r_0^2 - d(\beta(r_0), p)^2 = r_0^2 - \mathrm{L}(\gamma_{r_0})^2,$$

we conclude that $\phi(r)$ is convex. □

REMARK 1.119. *In general, if $\phi : (a,b) \to \mathbb{R}$ is a Lipschitz function such that for all $r_0 \in (a,b)$ there exists a C^2 function $\psi_{r_0}(r)$ defined in a neighborhood of r_0 with $\psi_{r_0}(r) \leq \phi(r)$, $\psi_{r_0}(r_0) = \phi(r_0)$ and $\left.\frac{d^2}{dr^2}\right|_{r=r_0} \psi_{r_0}(r) \geq 0$, then ϕ is convex.*

Using Lemma 1.118 one can give an elementary proof ([**234**]) of the Toponogov comparison theorem [**492**], [**3**] (see [**103**] for a different proof of the more general version).

THEOREM 1.120 (Toponogov - sect ≥ 0). *Let (M^n, g) be a complete Riemannian manifold with nonnegative sectional curvature and $\alpha : [0, A] \to M^n$ be a unit speed minimal geodesic joining p to q. If $\beta : [0, B] \to M^n$ is a unit speed geodesic with $\beta(0) = q$ and if $\theta \in [0, \pi]$ is the angle between $\dot{\beta}(0)$ and $-\dot{\alpha}(A)$ (so that $\cos\theta = \left\langle \dot{\beta}(0), -\dot{\alpha}(A) \right\rangle$), then*

$$d(\beta(r), p)^2 \leq r^2 + A^2 - 2Ar\cos\theta$$

for all $r > 0$. In particular of course,

$$d(\beta(B), p)^2 \leq A^2 + B^2 - 2AB\cos\theta. \tag{1.131}$$

By the law of cosines, equality is attained for euclidean space. That is, the RHS *of (1.131) is the length squared of the side in the corresponding euclidean triangle with the same A, B and θ.*

PROOF. For $\varepsilon > 0$, let

$$f_\varepsilon(r) \doteqdot r^2 - d(\beta(r), p)^2 + A^2 - 2Ar\cos\theta + \varepsilon r.$$

By the previous lemma, f_ε is convex. We also have $f_\varepsilon(0) = 0$ and by a first variation argument, $f_\varepsilon(r) > 0$ for $r > 0$ and small enough, depending on ε. Note that at a point where the distance function to p is smooth, we actually have

$$f_\varepsilon'(0) = -2d(\beta(0), p)\left\langle \dot{\beta}(0), \dot{\alpha}(A) \right\rangle - 2A\cos\theta + \varepsilon = \varepsilon > 0.$$

Since f_ε is convex, we conclude that $f_\varepsilon(r) > 0$ for all $r > 0$. In particular, $\lim_{\varepsilon\to 0} f_\varepsilon(r) \geq 0$ for all $r > 0$, which proves the theorem. □

EXERCISE 1.121. *Show that given $\varepsilon > 0$, indeed $f_\varepsilon(r) > 0$ for $r > 0$ and small enough.*

SOLUTION. Let $\alpha_r : [0, A] \to M^n$, $r \in [0, r_0]$, be a 1-parameter family of paths with $\alpha_r(0) = p$, $\alpha_r(A) = \beta(r)$ and $\alpha_0 = \alpha$. Consider the smooth barrier function

$$h_\varepsilon(r) \doteqdot r^2 - \mathrm{L}(\alpha_r)^2 + A^2 - 2Ar\cos\theta + \varepsilon r.$$

Clearly $f_\varepsilon(r) \geq h_\varepsilon(r)$ and $h_\varepsilon(0) = 0$ since α is minimal. We compute

$$h_\varepsilon'(0) = -2A\cos\theta - 2A \left.\frac{d}{dr}\right|_{r=0} \mathrm{L}(\alpha_r) + \varepsilon = \varepsilon > 0$$

since $\left.\frac{d}{dr}\right|_{r=0} \mathrm{L}(\alpha_r) = \left\langle \dot{\beta}(0), \dot{\alpha}(A) \right\rangle = -\cos\theta$ by the first variation formula (1.117). Thus $f_\varepsilon(r) \geq h_\varepsilon(r) > 0$ for $r > 0$ sufficiently small.

6.2. Long geodesics. Let $\gamma : [0, \bar{s}] \to M^n$ be a unit speed geodesic in a Riemannian n-manifold with $\mathrm{Rc} \leq (n-1) K$ in $B(\gamma(0), r)$ and $B(\gamma(\bar{s}), r)$ where $K > 0$ and $2r < \bar{s}$. Let $\{E_i\}_{i=1}^{n-1}$ be a parallel orthonormal frame along γ perpendicular to $\dot{\gamma}$. By the second variation of arc length formula (1.120) we have:

$$
\begin{aligned}
0 &\leq \sum_{i=1}^{n-1} \int_0^{\bar{s}} \left(\left| (\nabla_{\dot{\gamma}} (\varphi E_i))^{\perp} \right|^2 - \langle \mathrm{Rm}(\varphi E_i, \dot{\gamma}) \dot{\gamma}, \varphi E_i \rangle \right) ds \\
&= \int_0^{\bar{s}} \left((n-1) \left(\frac{d\varphi}{ds} \right)^2 - \varphi^2 \, \mathrm{Rc}(\dot{\gamma}, \dot{\gamma}) \right) ds
\end{aligned}
$$

for any function $\varphi : [0, \bar{s}] \to \mathbb{R}$. Now consider the piecewise smooth function

$$
\varphi(s) \doteqdot \begin{cases} \dfrac{s}{r} & \text{if } 0 \leq s \leq r \\ 1 & \text{if } r < s < \bar{s} - r \\ \dfrac{\bar{s} - s}{r} & \text{if } \bar{s} - r \leq s \leq \bar{s} \end{cases} .
$$

We then have $\left| \frac{d\varphi}{ds}(s) \right| = \frac{1}{r}$ for $s \in [0, r]$ and $s \in [\bar{s} - r, \bar{s}]$, and

$$
\begin{aligned}
\int_\gamma \mathrm{Rc}(\dot{\gamma}, \dot{\gamma}) \, ds &\leq \frac{2(n-1)}{r} + \int_0^r \left(1 - \varphi^2\right) \mathrm{Rc}(\dot{\gamma}, \dot{\gamma})_+ \, ds \\
&\quad + \int_{\bar{s} - r}^{\bar{s}} \left(1 - \varphi^2\right) \mathrm{Rc}(\dot{\gamma}, \dot{\gamma})_+ \, ds \\
&\leq 2(n-1) \left(\frac{1}{r} + Kr \right)
\end{aligned}
$$

where $\mathrm{Rc}(\dot{\gamma}, \dot{\gamma})_+ \doteqdot \max\{\mathrm{Rc}(\dot{\gamma}, \dot{\gamma}), 0\}$.

A fundamental result about noncompact manifolds with positive sectional curvature is the following [**240**] (see also p. 312 of [**153**] for an exposition).

THEOREM 1.122 (Gromoll-Meyer). *If (M^n, g) is a complete Riemannian manifold with positive sectional curvature bounded above by K, then*

$$
\mathrm{inj}(M^n, g) \geq \pi / \sqrt{K}.
$$

The Gromoll-Meyer Theorem is important in the study of singularity formulation of solutions to the Ricci flow in dimension 3. The reason is because singularity models of finite time singular solutions have nonnegative sectional curvature (see §3.3.2 of Chapter 5).

7. Green's function

In euclidean space the Green's function, or fundamental solution of the Laplace equation, is given by:

$$G(x,y) \doteqdot \begin{cases} \dfrac{1}{n(n-2)\,\omega_n} |x-y|^{2-n} & \text{if } n \geq 3 \\ -\dfrac{1}{2\pi} \log|x-y| & \text{if } n = 2 \end{cases},$$

where ω_n is the volume of the unit euclidean n-ball. Note that $G(x,y) > 0$ for $n \geq 3$ and $G(x,y)$ changes sign for $n = 2$. One calculates that for any function φ with compact support in $\mathbb{R}^n$ we have (see p. 17-18 of [**227**])

$$\varphi(x) = -\int_{\mathbb{R}^n} G(x,y)\,\Delta\varphi(y)\,dy.$$

Let

$$h(x,y,t) \doteqdot (4\pi t)^{-n/2} \exp\left(-\frac{|x-y|^2}{4t}\right)$$

be the euclidean heat kernel. Recall that the Gamma function is defined by

$$\Gamma(x) \doteqdot \int_0^\infty t^{x-1} e^{-t} dt$$

for $x > 0$, and satisfies $\Gamma(x+1) = x\Gamma(x)$. The volume of the unit n-ball is given by

$$\omega_n = \frac{\pi^{n/2}}{\Gamma\left(\frac{n}{2}+1\right)}.$$

Hence making the change of variables $u = \frac{|x-y|^2}{4t}$, we have for $n \geq 3$

$$\begin{aligned} \int_0^\infty h(x,y,t)\,dt &= \frac{1}{4}\pi^{-n/2} |x-y|^{2-n} \int_0^\infty u^{(n-4)/2} e^{-u} du \\ &= \frac{1}{n(n-2)\,\omega_n} |x-y|^{2-n} = G(x,y). \end{aligned}$$

Given a complete, noncompact Riemannian manifold (M^n, g), a nonnegative **Green's function** is a smooth function

$$G : M^n \times M^n - \Delta(M) \to \mathbb{R},$$

where $\Delta(M) \doteqdot \{(x,x) : x \in M^n\}$, such that

(1) $G(x,y) = G(y,x) \geq 0$
(2) Given any $y \in M^n$,

$$\Delta_x G(x,y) = 0 \quad \text{for all } x \in M^n - \{y\}.$$

(3) Given $y \in M^n$, for x close to y we have

$$G(x,y) = \begin{cases} (1+o(1))\,d(x,y)^{2-n} & \text{if } n > 2 \\ -(1+o(1)) \log d(x,y) & \text{if } n = 2 \end{cases}.$$

On a complete Riemannian manifolds there always exists a Green's function. There is also a characterization of when there exists a positive Green's function.

THEOREM 1.123 (Li-Tam [**342**]). *For any complete Riemannian manifold (M^n, g) there exists a Green's function $G(x,y)$ which is smooth on $M \times M - D$, where D is the diagonal. $G(x,y)$ can be taken to be positive if and only if there exists a positive superharmonic function f on $M - \Omega$, where Ω is some compact domain, satisfying*

$$\liminf_{x\to\infty} f(x) < \inf_{x\in\partial\Omega} f(x).$$

If (M^n, g) is a complete Riemannian manifold, then we can define a Green's function $G_0(x,y)$ in terms of the heat kernel by:

$$G_0(x,y) \doteqdot \int_0^\infty H(x,y,t)\,dt$$

whenever the RHS converges.

LEMMA 1.124. $G_0(x,y) > 0$ *and*

$$\Delta_x G_0(x,y) = -\delta_y(x).$$

8. Comparison theory for the heat kernel

The **fundamental solution of the heat equation** (or **heat kernel**) on $\mathbb{R}^n$ is

$$h(x,y,t) \doteqdot (4\pi t)^{-n/2} \exp\left\{-\frac{|x-y|^2}{4t}\right\}. \tag{1.132}$$

The heat kernel is symmetric in x and y and it is easy to see that

$$\left(\frac{\partial}{\partial t} - \Delta_x\right) h(\cdot, y, \cdot) = 0.$$

We also have $\lim_{t\to 0} h(\cdot, y, t) = \delta_y$, that is,

$$\lim_{t\to 0}\int_{\mathbb{R}^n} h(x,y,t) f(x)\,dx = f(y)$$

for all $f \in C^0(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$. Analogous to the Laplacian and volume comparison theorem, we have the following [**110**].

THEOREM 1.125 (Cheeger-Yau). *If (M^n, g) is a complete Riemannian manifold with* $\mathrm{Rc} \geq -(n-1)H$ *for some $H \in \mathbb{R}$, then the fundamental solution $h(x,y,t)$ of the heat equation, that is, the minimum positive heat kernel on M^n satisfies*

$$h(x,y,t) \geq h_{n,H}(d_g(x,y), t)$$

where $h_{n,H}$ is the heat kernel of the simply connected, complete, n-dimensional manifold of constant sectional curvature H.

Here we have used the fact that at each time $h_{n,H}$ is a radial function. Taking $H = 0$, we have the following.

COROLLARY 1.126. *If (M^n, g) is a complete Riemannian manifold with nonnegative Ricci curvature, then the fundamental solution $h(x, y, t)$ of the heat equation on M^n satisfies*

$$h(x, y, t) \geq (4\pi t)^{-n/2} e^{-d(x,y)^2/4t}. \tag{1.133}$$

In fact, Cheeger and Yau prove comparison theorems for fundamental solutions on geodesic balls with Dirichlet and Neumann boundary conditions. Inequality (1.133) is also a consequence of the Li-Yau differential Harnack inequality for positive solutions of the heat equation (see (8.8)).

For the hyperbolic plane $\mathbb{H}^2$ (the complete surface with constant curvature -1), we have (see p. 246 of [**97**])

$$h_{2,-1}(x, y, t) = \frac{\sqrt{2}e^{-t/4}}{(4\pi t)^{3/2}} \int_{d(x,y)}^{\infty} \frac{se^{-s^2/4t}}{\sqrt{\cosh s - \cosh d(x, y)}} ds$$

for $x, y \in \mathbb{H}^2$ and $t > 0$. More generally, we have the following explicit formulas (see Grigor'yan and Noguchi [**239**].)

THEOREM 1.127. *The heat kernel of hyperbolic space $\mathbb{H}^n$ is given in even dimensions by:*

$$\begin{aligned} &h_{2(m+1),-1}(x, y, t) \\ &= \left(\frac{-1}{2\pi}\right)^m \frac{\sqrt{2}e^{-(2m+1)^2 t/4}}{(4\pi t)^{3/2}} \left(\frac{1}{\sinh r}\frac{\partial}{\partial r}\right)^m \left(\int_r^{\infty} \frac{se^{-s^2/4t}}{\sqrt{\cosh s - \cosh r}} ds\right) \end{aligned}$$

and in odd dimensions by:

$$h_{2m+1,-1}(x, y, t) = \left(\frac{-1}{2\pi}\right)^m \frac{e^{-m^2 t}}{(4\pi t)^{1/2}} \left(\frac{1}{\sinh r}\frac{\partial}{\partial r}\right)^m \left(e^{-r^2/4t}\right)$$

where $r \doteqdot d(x, y)$.

EXERCISE 1.128. *Using the fact that $\Delta = \frac{\partial^2}{\partial r^2} + \coth r \frac{\partial}{\partial r}$ acting on radial functions on $\mathbb{H}^2$ (see (1.101)), show that $h_{2,-1}(x, y, t)$ is a solution of the heat equation on $\mathbb{H}^2$. Furthermore, show that it is a fundamental solution.*

Li and Yau [**346**] have proved the following lower bound for the heat kernel (see also p. 333 of [**97**] for this statement).

THEOREM 1.129. *If (M^n, g) is a complete Riemannian manifold with $\mathrm{Rc} \geq -(n-1)H$ where $H \leq 0$, then for any $\varepsilon \in (0, 4)$ there exists $c(n, \varepsilon) > 0$ and $C(n) < \infty$ such that*

$$h(x, y, t) \geq \frac{c(n, \varepsilon)}{\mathrm{Vol}\left(B\left(x, \sqrt{t}\right)\right)^{\frac{1}{2}} \mathrm{Vol}\left(B\left(y, \sqrt{t}\right)\right)^{\frac{1}{2}}} \exp\left\{-\frac{d(x, y)^2}{(4-\varepsilon)t} + C(n)\varepsilon H t\right\}.$$

As one might expect, $\lim_{\varepsilon \to 0} c(n, \varepsilon) = 0$.

REMARK 1.130. *See §8.1 for a related discussion of positive solutions of the heat equation. Chavel's book* [**97**] *is a good source for the material on the heat equation (see especially Chapters 6-8). See §12.4 of* [**97**] *for a heuristic derivation of (1.132) as the euclidean heat kernel.*

Comparison theory for the heat kernel is related to volume comparison theory (§4) and also Perelman's comparison theory for the Ricci flow (see Volume 2.)

9. Parametrix for the heat equation

We now consider the heat kernel associated to a 1-parameter family of metrics $g(t)$, $t \in [0,T)$, on a manifold M^n. In this section we sketch the ideas behind obtaining an asymptotic expansion for the heat kernel associated to an evolving metric. The idea we follow is to modify the techniques in the fixed metric case. Our main interest will be to apply this to the case where the metrics are evolving by the Ricci flow. We hope the interested reader will pursue these ideas since we are not aware of the variable metric case being considered in the literature.

Let $u(x,y,t) \doteqdot (4\pi t)^{-n/2} \exp\left(-\frac{d_{g(0)}^2(x,y)}{4t}\right)$, which in the case of euclidean space is the heat kernel. For each $N \in \mathbb{N}$ we shall construct a function of the form:

$$h_N(x,y,t) \doteqdot u(x,y,t) \sum_{k=0}^{N} \phi_k(x,y)\, t^k$$

where $\phi_k(x,y)$, to be defined for all $x,y \in M^n$ such that $d_{g(0)}(x,y) < \operatorname{inj}(g(0))$, will be chosen so that

$$\left(\Delta_{g(t)} - \frac{\partial}{\partial t}\right) h_N = O\left(t^N\right)$$
$$\lim_{t\to 0_+} h_N(x,y,t) = \delta_x(y).$$

Here $O\left(t^N\right)$ means that there exists $C < \infty$ such that

$$\left|\left(\Delta_{g(t)} - \frac{\partial}{\partial t}\right) h_N(x,y,t)\right| \leq C t^N$$

for all $x,y \in M^n$ with $d_{g(0)}(x,y) < \operatorname{inj}(g(0))$ and $t < 1$. For each $(x,y) \in M^n \times M^n$, we are writing the ratio $\frac{h_N(x,y,t)}{u(x,y,t)}$ as a finite power series in t and we want h_N to be a fundamental solution to the heat equation up to order N in time.

Recall that the Laplacian of a radial function $f = f(r)$ is given inside the cut locus $\operatorname{Cut}(x)$ by (1.87):

$$\Delta f = \frac{d^2 f}{dr^2} + \frac{\partial}{\partial r} \log \sqrt{\det g} \frac{df}{dr}$$

where $r \doteqdot d_g(x,y)$. Since $\left(\frac{\partial^2}{\partial r^2} + \frac{n-1}{r}\frac{\partial}{\partial r} - \frac{\partial}{\partial t}\right) u = 0$, we have

$$\left(\Delta_{g(0)} - \frac{\partial}{\partial t}\right) u = \frac{\partial \log \alpha(0)}{\partial r}\frac{du}{dr} = -\frac{r}{2t}\frac{\partial \log \alpha(0)}{\partial r} u$$

where $\alpha(0) \doteqdot \sqrt{\det g(0)} \Big/ r^{n-1}$. Note that $\alpha(0)$ is bounded if r is bounded and

$$\frac{\partial \log \alpha(0)}{\partial r} = \frac{\partial}{\partial r} \log \sqrt{\det g(0)} - \frac{n-1}{r} = o(1) \tag{1.134}$$

for r small.

REMARK 1.131. *In the following we will exhibit in our notation the dependence on time of the metric and its associated quantities and operators when necessary for clarity.*

We assume the metric at time t may be written as

$$g(t) = g(0) + \sum_{m=1}^{N} h_m t^m + O\left(t^{N+1}\right),$$

where

$$h_m \doteqdot \frac{1}{m!}\frac{\partial^m g}{\partial t^m}(0)$$

which are symmetric 2-tensors. For example, if $g(t)$ is a solution of the Ricci flow, then $h_1 = -2\operatorname{Rc}(g(0))$. Let

$$g(t)^{ij} \doteqdot g(0)^{ij} + \sum_{m=1}^{N} w_m^{ij} t^m + O\left(t^{N+1}\right),$$

$$\log \alpha(t) = \log \frac{\sqrt{\det g(t)}}{r^{n-1}} \doteqdot \log \alpha(0) + \sum_{k=1}^{N} z_k t^k + O\left(t^{N+1}\right)$$

and

$$\Gamma(t)_{ij}^k \doteqdot \Gamma(0)_{ij}^k + \sum_{p=1}^{N} (A_p)_{ij}^k t^p + O\left(t^{N+1}\right)$$

where w_m are $(0,2)$-tensors, z_k are functions, and A_p are $(2,1)$-tensors. We have (see (2.14))

$$w_1^{ij} = -g(0)^{ik}(h_1)_{k\ell}\, g(0)^{\ell j}, \quad z_1 = \frac{1}{2} g(0)^{ij}(h_1)_{ij}$$

and (see (2.18))

$$(A_1)_{ij}^k = \frac{1}{2}\left(\nabla_i (h_1)_j^k + \nabla_j (h_1)_i^k - \nabla^k (h_1)_{ij}\right).$$

where the covariant derivatives and raising of indices are with respect to $g(0)$. Note that for a solution of the Ricci flow

$$w_1^{ij} = 2g^{ik} R_{k\ell} g^{\ell j}, \quad z_1 = -R$$

and

$$(A_1)_{ij}^k = -\nabla_i R_j^k - \nabla_j R_i^k + \nabla^k R_{ij}$$

where all of the quantities on the RHS are with respect to $g = g(0)$.

The Laplacian at time t is:

$$\begin{aligned}
\Delta_{g(t)} &= g(t)^{ij}\left(\frac{\partial^2}{\partial x^i \partial x^j} - \Gamma(t)_{ij}^k \frac{\partial}{\partial x^k}\right) \\
&= \left(g(0)^{ij} + \sum_{m=1}^N t^m w_m^{ij} + O\left(t^{N+1}\right)\right) \\
&\times \left(\frac{\partial^2}{\partial x^i \partial x^j} - \left(\Gamma(0)_{ij}^k + \sum_{p=1}^N t^p (A_p)_{ij}^k + O\left(t^{N+1}\right)\right)\frac{\partial}{\partial x^k}\right) \\
&= \Delta + \sum_{m=1}^N t^m w_m^{ij} \nabla_i \nabla_j - \sum_{p=1}^N t^p g^{ij} (A_p)_{ij}^k \nabla_k \\
&- \sum_{q=2}^{2N} \sum_{m+p=q} t^q \left(w_m^{ij} (A_p)_{ij}^k\right) \nabla_k + O\left(t^{N+1}\right) \\
&\doteqdot \Delta + \sum_{m=1}^N t^m \left(w_m^{ij} \nabla_i \nabla_j + v_m^k \nabla_k\right) + O\left(t^{N+1}\right),
\end{aligned}$$

where again the RHS is with respect to $g(0)$ and last equality defines the vector fields $v_m = v_m^k \frac{\partial}{\partial x^k}$. In particular

$$\Delta_{g(t)} = \Delta + t\left(w_1^{ij} \nabla_i \nabla_j - g^{ij} (A_1)_{ij}^k \nabla_k\right) + O\left(t^2\right).$$

Let $\psi_N \doteqdot \sum_{k=0}^N \phi_k(x,y) t^k$ so that $h_N = u\psi_N$. We compute

$$\left(\Delta_{g(t)} - \frac{\partial}{\partial t}\right) h_N = \psi_N \left(\Delta_{g(t)} - \frac{\partial}{\partial t}\right) u + u \left(\Delta_{g(t)} - \frac{\partial}{\partial t}\right) \psi_N + 2 \langle du, d\psi_N \rangle_{g(t)}.$$

Since $\langle du, d\psi_N \rangle_g = g^{ij} \frac{\partial u}{\partial x^i} \frac{\partial \psi_N}{\partial x^j}$ we have

$$\begin{aligned}
\langle du, d\psi_N \rangle_{g(t)} &= \langle du, d\psi_N \rangle_{g(0)} + \sum_{m=1}^N w_m^{ij} t^m \frac{\partial u}{\partial x^i} \frac{\partial \psi_N}{\partial x^j} \\
&= \frac{\partial u}{\partial r}\frac{\partial \phi_0}{\partial r} + \sum_{k=1}^N t^k \frac{\partial u}{\partial r}\frac{\partial \phi_k}{\partial r} + \sum_{m=1}^N \sum_{k=0}^N t^{m+k} w_m^{ij} \frac{\partial u}{\partial x^i}\frac{\partial \phi_k}{\partial x^j}
\end{aligned}$$

since u is a radial function at $t = 0$. From this we obtain

$$\begin{aligned}\left(\Delta_{g(t)} - \frac{\partial}{\partial t}\right) h_N &= u \sum_{k=0}^{N} \left(\Delta_{g(t)}\phi_k - \frac{r}{2t}\frac{\partial \log \alpha}{\partial r}\phi_k - \frac{r}{t}\frac{\partial \phi_k}{\partial r} - \frac{k}{t}\phi_k\right) t^k \\ &+ 2\sum_{m=1}^{N}\sum_{k=0}^{N} t^{m+k} w_m^{ij} \frac{\partial u}{\partial x^i}\frac{\partial \phi_k}{\partial x^j} \\ &= t^N u \Delta_{g(t)}\phi_N - \frac{r}{t}u\left(\frac{1}{2}\frac{\partial \log \alpha}{\partial r}\phi_0 + \frac{\partial \phi_0}{\partial r}\right) \\ &+ \frac{u}{t}\sum_{k=1}^{N}\left(\Delta_{g(t)}\phi_{k-1} - \left(k + \frac{r}{2}\frac{\partial \log \alpha}{\partial r}\right)\phi_k - r\frac{\partial \phi_k}{\partial r}\right) t^k \\ &- u\frac{r}{t}\sum_{m=1}^{N}\sum_{k=0}^{N} t^{m+k} w_m^{ij}\frac{\partial r}{\partial x^i}\frac{\partial \phi_k}{\partial x^j}\end{aligned}$$

Now we apply the expansion for $\Delta_{g(t)}$ and α to get

$$\begin{aligned}\left(\Delta_{g(t)} - \frac{\partial}{\partial t}\right) h_N &= -\frac{r}{t}u\left(\frac{1}{2}\frac{\partial \log \alpha(0)}{\partial r}\phi_0 + \frac{\partial \phi_0}{\partial r}\right) - \frac{r}{2t}u\sum_{k=1}^{N} t^k \frac{\partial z_k}{\partial r}\phi_0 \\ &+ \frac{u}{t}\sum_{k=1}^{N} t^k\left(\Delta\phi_{k-1} + \sum_{m=1}^{N} t^m\left(w_m^{ij}\nabla_i\nabla_j + v_m^{\ell}\nabla_\ell\right)\phi_{k-1}\right) \\ &+ \frac{u}{t}\sum_{k=1}^{N}\left(-\left(k + \frac{r}{2}\frac{\partial \log \alpha(0)}{\partial r}\right)\phi_k - r\frac{\partial \phi_k}{\partial r}\right) t^k \\ &- \frac{r}{2t}u\sum_{m=1}^{N} t^m \frac{\partial z_m}{\partial r}\sum_{k=1}^{N}\phi_k t^k - u\frac{r}{t}\sum_{m=1}^{N}\sum_{k=0}^{N} t^{m+k} w_m^{ij}\frac{\partial r}{\partial x^i}\frac{\partial \phi_k}{\partial x^j} \\ &+ O\left(t^N\right).\end{aligned}$$

We write this as

$$\begin{aligned}
&\left(\Delta_{g(t)} - \frac{\partial}{\partial t}\right) h_N \\
&= -\frac{r}{t} u \left(\frac{1}{2} \frac{\partial \log \alpha(0)}{\partial r} \phi_0 + \frac{\partial \phi_0}{\partial r}\right) \\
&+ \frac{u}{t} \sum_{k=1}^{N} t^k \left(-r \frac{\partial \phi_k}{\partial r} - \left(k + \frac{r}{2} \frac{\partial \log \alpha(0)}{\partial r}\right) \phi_k + \Delta \phi_{k-1} - \frac{r}{2} \frac{\partial z_k}{\partial r} \phi_0\right) \\
&+ \frac{u}{t} \sum_{k=2}^{2N} \sum_{\substack{m+p=k, \\ m,p \geq 1}} \left(\left(w_m^{ij} \nabla_i \nabla_j + v_m^{\ell} \nabla_{\ell}\right) \phi_{p-1} - \frac{r}{2} \frac{\partial z_m}{\partial r} \phi_p\right) t^k \\
&- u \frac{r}{t} \sum_{k=1}^{2N} \sum_{\substack{m+p=k \\ m \geq 1,\ p \geq 0}} t^k w_m^{ij} \frac{\partial r}{\partial x^i} \frac{\partial \phi_p}{\partial x^j} \\
&+ O\left(t^N\right).
\end{aligned}$$

When $k = 0$ we require ϕ_0 to be a solution of the ODE

$$\frac{\partial}{\partial r} \phi_0 + \frac{1}{2} \frac{\partial \log \alpha(0)}{\partial r} \phi_0 = 0$$

so that

$$\phi_0(x, y) = \alpha(0)^{-1/2}(x, y).$$

When $k = 1$ we impose the ODE

$$r \frac{\partial \phi_1}{\partial r} + \left(1 + \frac{r}{2} \frac{\partial \log \alpha(0)}{\partial r}\right) \phi_1 = \Delta \phi_0 - r w_1^{ij} \frac{\partial r}{\partial x^i} \frac{\partial \phi_0}{\partial x^j} - \frac{r}{2} \frac{\partial z_1}{\partial r} \phi_0. \tag{1.135}$$

For $2 \leq k \leq N$, define ϕ_k inductively by the following ODE along rays emanating from x

$$\begin{aligned}
&r \frac{\partial \phi_k}{\partial r} + \left(k + \frac{r}{2} \frac{\partial \log \alpha(0)}{\partial r}\right) \phi_k \\
&\doteqdot \Phi_k = \Delta \phi_{k-1} - \frac{r}{2} \frac{\partial z_k}{\partial r} \phi_0 - r w_k^{ij} \frac{\partial r}{\partial x^i} \frac{\partial \phi_0}{\partial x^j} \\
&+ \sum_{\substack{m+p=k \\ m,p \geq 1}} \left(\left(w_m^{ij} \nabla_i \nabla_j + v_m^{\ell} \nabla_{\ell}\right) \phi_{p-1} - \frac{r}{2} \frac{\partial z_m}{\partial r} \phi_p - r w_m^{ij} \frac{\partial r}{\partial x^i} \frac{\partial \phi_p}{\partial x^j}\right).
\end{aligned} \tag{1.136}$$

for $k = 0, \ldots, N$ where $\phi_{-1} \doteqdot 0$ and $\phi_0(x, x) \doteqdot 1$. Then

$$\left(\Delta - \frac{\partial}{\partial t}\right) h_N = O\left(t^N\right).$$

Now (1.136) implies for $k \geq 1$

$$\frac{\partial}{\partial r}\left(r^k \alpha^{1/2} \phi_k\right) = r^{k-1} \alpha^{1/2} \Phi_k$$

so that

$$\phi_k(x,y) = r^{-k}\alpha^{-1/2}\int_0^r \bar{r}^{k-1}\alpha^{1/2}\Phi_k d\bar{r}.$$

where the integration is along the unit speed minimal geodesic emanating from x and and ending at y.

We have

$$\frac{\partial^2}{\partial r^2}\log\frac{\sqrt{\det g_{ij}}}{r^{n-1}} = \frac{\partial H}{\partial r} + \frac{n-1}{r^2} = -\operatorname{Rc}\left(\frac{\partial}{\partial r},\frac{\partial}{\partial r}\right) - |h|^2 + \frac{n-1}{r^2}.$$

Since $\lim_{r\to 0_+}\left(-|h|^2 + \frac{n-1}{r^2}\right) = 0$, for any unit vector $V \in T_pM^n$, by taking limit along the geodesic $\gamma_V : [0,L) \to M^n$ emanating from p with $\dot{\gamma}_V(0) = V$, we have

$$\begin{aligned}(\nabla_V\nabla_V \log\alpha(0))(p) &= \lim_{r\to 0_+}\frac{\partial^2}{\partial r^2}\log\frac{\sqrt{\det g_{ij}}}{r^{n-1}} \\ &= -\operatorname{Rc}\left(\frac{\partial}{\partial r},\frac{\partial}{\partial r}\right) = -\operatorname{Rc}(V,V).\end{aligned}$$

Hence

$$(\Delta\log\alpha(0))(p) = -R(p).$$

From taking the limit as $r\to 0_+$ in (1.134) we have

$$|\nabla\log\alpha(0)|^2(p) = 0.$$

Since $\alpha(0)(p) = 1$, we conclude that

$$(\Delta\alpha(0))(p) = -R(p)$$

and

$$\Delta\phi_0(p) = \frac{1}{2}R(p).$$

The **parametrix** is defined by

$$p_N(x,y,t) \doteqdot \eta\left(d_{g(0)}(x,y)\right)h_N(x,y,t)$$

where $\eta : M^n \to [0,1]$ is a smooth radial cutoff function with $\eta(y) = 1$ if $d_{g(0)}(x,y) \le \frac{1}{2}\operatorname{inj}(g(0))$ and $\eta(y) = 0$ if $d_{g(0)}(x,y) \ge \operatorname{inj}(g(0))$.

The asymptotics of the heat kernel on Riemannian manifolds are related to the determinant of the Laplacian (see §11) and index theorems. For time-dependent metrics, the asymptotics of the heat kernel are related to Perelman's Harnack type inequality for the adjoint heat equation (see Volume 2.)

10. Eigenvalues and eigenfunctions of the Laplacian

Let (M^n, g) be a complete Riemannian manifold. Let $L^2(M^n, g)$ denote the Hilbert space of measurable square integrable functions with the L^2-**inner product**:

$$\langle f,h\rangle_{L^2} = \int_{M^n} fh\,d\mu.$$

As usual, by a **measurable function** we mean an equivalence class of measurable functions under the equivalence relation of differing on a set of measure zero. When we say that a measurable function satisfies a pointwise property such as being in C^∞ or having L^∞ (sup) norm bounded by some constant, we mean that some function in the equivalence class satisfies the property. Given C^∞ functions f and h, let

$$\langle f, h \rangle_{W^{1,2}} = \langle f, h \rangle_{L^2} + \int_{M^n} \langle df, dh \rangle \, d\mu.$$

Let $W^{1,2}$ denote the Hilbert space completion of $C^\infty(M^n)$ with respect to the inner product $\langle \cdot\, , \cdot \rangle_{W^{1,2}}$.

The eigenfunction equation is

$$\Delta \phi + \lambda \phi = 0 \tag{1.137}$$

where $\phi : M^n \to \mathbb{R}$ is called the **eigenfunction** and $\lambda \in \mathbb{R}$ is called the **eigenvalue**. If M^n is closed, then assuming $\phi \not\equiv 0$ we have

$$\lambda = \frac{\int_{M^n} |\nabla \phi|^2 \, d\mu}{\int_{M^n} \phi^2 d\mu},$$

which is obtained from multiplying (1.137) by ϕ and integrating by parts. On a *closed* manifold, the set of eigenvalues

$$0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_k \leq \cdots$$

is discrete. This may be seen as follows. The operator $-\Delta + 1$ is invertible and its inverse $(-\Delta + 1)^{-1} : L^2(M^n) \to W^{2,2}(M^n) \hookrightarrow L^2(M^n)$ is a compact operator by elliptic regularity. Thus the eigenvalues

$$\cdots \leq \mu_k \leq \cdots \leq \mu_0 = 1$$

of $(-\Delta + 1)^{-1}$ are discrete and $\lim_{k\to\infty} \mu_k = 0$. From $\lambda_k = \frac{1-\mu_k}{\mu_k}$ we see that $\lim_{k\to\infty} \lambda_k = \infty$. The **Weyl asymptotic formula** says that

$$\lambda_k \sim c_n \operatorname{Vol}(g)^{-2/n} k^{2/n} \tag{1.138}$$

where $c_n = 4\pi^2 \omega_n^{-2/n}$ and ω_n is the volume of the unit euclidean n-ball ($\sim$ means the ratio of the RHS to the LHS approaches 1 as $k \to \infty$.) See §11.1 for a sketch of the proof of (1.138). For each λ_k we may choose the corresponding eigenfunction ϕ_k so that $\{\phi_k\}_{k=1}^\infty$ is orthonormal. Then $\{\phi_k\}_{k=1}^\infty$ is a basis for $L^2(M^n)$. For any function $f \in L^2(M^n)$ we have

$$f = \sum_{k=1}^\infty \langle f, \phi_k \rangle_{L^2} \phi_k$$

and

$$\|f\|_{L^2}^2 = \sum_{k=1}^\infty \langle f, \phi_k \rangle_{L^2}^2 .$$

The **Rayleigh principle** says that

$$\lambda_k = \inf_{f \perp H_{k-1}} \frac{\int_{M^n} |\nabla f|^2 \, d\mu}{\int_{M^n} f^2 d\mu}$$

where $H_{k-1} \doteqdot \operatorname{span}\{\phi_0, \ldots, \phi_{k-1}\} \subset L_1^2(M^n)$. In particular

$$\lambda_1 = \inf \left\{ \frac{\int_{M^n} |\nabla f|^2 \, d\mu}{\int_{M^n} f^2 d\mu} : f \in L_1^2(M^n), \ \int_{M^n} f d\mu = 0 \right\}.$$

10.1. Lower bound for the first eigenvalue of the Laplacian. A beautiful eigenvalue lower bound, depending on a gradient estimate is the following.

THEOREM 1.132 (Li-Yau). *Let (M^n, g) be a closed Riemannian manifold with nonnegative Ricci curvature. Then*

$$\lambda_1 \geq \frac{\pi^2}{4 \operatorname{diam}(g)^2}$$

where $\operatorname{diam}(g)$ *is the diameter of* (M^n, g).

PROOF. Suppose $\Delta u + \lambda u = 0$ and define f by

$$f = |\nabla u|^2 + (\lambda + \varepsilon) u^2$$

where $\varepsilon > 0$. We compute

$$\frac{1}{2}\Delta f = (u_{ij})^2 + u_i u_{ijj} + (\lambda + \varepsilon) u \Delta u + (\lambda + \varepsilon) |\nabla u|^2.$$

Now

$$u_{ijj} = u_{jij} = u_{jji} - R_{jijk} u_k = -\lambda u_i + R_{ik} u_k.$$

Therefore, using the eigenvalue equation, we have

$$\frac{1}{2}\Delta f = (u_{ij})^2 - \lambda (\lambda + \varepsilon) u^2 + \varepsilon |\nabla u|^2 + \operatorname{Rc}(\nabla u, \nabla u).$$

By the maximum principle, at a point x where f attains its maximum, $\nabla f(x) = 0$ and $\Delta f(x) \leq 0$. We have

$$\frac{1}{2} f_i = u_j u_{ji} + (\lambda + \varepsilon) u \cdot u_i.$$

Choose special coordinates so that at x, $u_i = 0$ for $i = 2, \ldots, n$. Then

$$0 = \frac{1}{2} f_1(x) = u_1 (u_{11} + (\lambda + \varepsilon) u)(x).$$

Therefore, if $\nabla u(x) \neq 0$, then

$$\begin{aligned} 0 &\geq \frac{1}{2}\Delta f \geq u_{11}(x)^2 - \lambda (\lambda + \varepsilon) u(x)^2 + \varepsilon |\nabla u|^2 (x) \\ &\geq \varepsilon (\lambda + \varepsilon) u(x)^2 + \varepsilon |\nabla u|^2 (x), \end{aligned}$$

which is a contradiction. Hence $\nabla u(x) = 0$ at a point x where f attains its maximum. Let $\mu \doteqdot \max_{y \in M} |u(y)|$. Then

$$f(y) \leq f(x) = (\lambda + \varepsilon) u(x)^2 \leq (\lambda + \varepsilon) \mu^2$$

for all $y \in M$. Thus, for all $y \in M$

$$|\nabla u|^2 (y) \leq (\lambda + \varepsilon) \left(\mu^2 - u(y)^2 \right).$$

Since the above inequality is true for all $\varepsilon > 0$, we have

$$|\nabla u|^2 \leq \lambda \left(\mu^2 - u^2 \right).$$

Now u is zero somewhere since

$$0 = \int_M (\Delta u + \lambda u) \, d\mu = \lambda \int_M u d\mu.$$

Let γ be a geodesic joining points x and y, where $u(x) = 0$ and $|u(y)| = \mu$. Then by the inequality above,

$$\begin{aligned} \operatorname{diam}(g) \sqrt{\lambda} &\geq \int_\gamma \sqrt{\lambda} ds \geq \int_\gamma \frac{|\nabla u|}{\sqrt{\mu^2 - u^2}} ds \\ &\geq \int_0^\mu \frac{du}{\sqrt{\mu^2 - u^2}} = \frac{\pi}{2}. \end{aligned}$$

The theorem follows. □

The above result extends to manifolds with boundary.

Theorem 1.133 (Li-Yau). *Suppose that (M^n, g) is compact with $\operatorname{Rc} \geq 0$ and boundary ∂M. Let λ_D and λ_N denote the first eigenvalues of the Laplacian with Dirichlet and Neumann boundary conditions, respectively.*

(1) *If the mean curvature H of ∂M is nonnegative, then*

$$\lambda_D \geq \frac{\pi^2}{4 \operatorname{diam}(g)^2}.$$

(2) *If ∂M is convex, that is, $\mathrm{II}_{ij} \geq 0$, then*

$$\lambda_N \geq \frac{\pi^2}{4 \operatorname{diam}(g)^2}.$$

Proof. Let $f = |\nabla u|^2 + (\lambda + \varepsilon) u^2$ as before. If the maximum of f is attained in the interior of M, then we can apply the arguments of the previous theorem to obtain the desired estimates. Therefore we suppose the maximum of f is attained on the boundary, say at $x \in \partial M$, and derive a contradiction. Choose an orthonormal frame field $\{e_i\}_{i=1}^n$ in a neighborhood of x in ∂M so that $e_n = \nu$ is the outward unit normal. The strong maximum principle implies

$$f_n(x) = \frac{\partial f}{\partial \nu}(x) > 0.$$

We claim that

$$\frac{1}{2}f_n = -Hu_n^2 - \sum_{j,k=1}^{n-1} h_{jk}u_ju_k. \tag{1.139}$$

Assuming (1.139), for the Dirichlet problem we have at x

$$0 < \frac{1}{2}f_n = -Hu_n^2$$

which is impossible. Similarly, for the Neumann problem, at x

$$0 < \frac{1}{2}f_n = -\sum_{j,k=1}^{n-1} h_{jk}u_ju_k$$

which is also impossible by our convexity assumption.

Now we prove (1.139). We have at a point on the boundary

$$\begin{aligned}\frac{1}{2}f_n &= \sum_{j=1}^{n} u_ju_{jn} + (\lambda+\varepsilon)\,u\cdot u_n\\ &= u_nu_{nn} + \sum_{j=1}^{n-1} u_ju_{jn}\end{aligned}$$

in either the case of the Dirichlet problem ($u=0$) or the Neumann problem ($u_n = 0$). Now for the Dirichlet problem $u = u_i = 0$ for $i \neq n$. Thus

$$\begin{aligned}u_{nn} &= \Delta u - \sum_{i=1}^{n-1} u_{ii} = -\lambda u - \sum_{i=1}^{n-1}\left(e_ie_iu - \left(\nabla_{e_i}e_i\right)u\right)\\ &= \sum_{i=1}^{n-1}\left\langle \nabla_{e_i}e_i, \nu\right\rangle u_n = -Hu_n.\end{aligned}$$

For the Neumann problem, $u_n = 0$. So in either case

$$u_nu_{nn} = -Hu_n^2.$$

We are left to compute $\sum_{j=1}^{n-1} u_ju_{jn}$. For $j \neq n$

$$\begin{aligned}u_{jn} = u_{nj} &= e_je_nu - \left(\nabla_{e_j}e_n\right)u\\ &= e_j\left(u_n\right) - \sum_{k=1}^{n-1}\omega_n^k\left(e_j\right)u_k\\ &= e_j\left(u_n\right) - \sum_{k=1}^{n-1}\mathrm{II}_{jk}\,u_k.\end{aligned}$$

Therefore

$$\begin{aligned}\sum_{j=1}^{n-1} u_j u_{jn} &= \sum_{j=1}^{n-1} u_j e_j (u_n) - \sum_{j,k=1}^{n-1} \mathrm{II}_{jk}\, u_j u_k \\ &= -\sum_{j,k=1}^{n-1} \mathrm{II}_{jk}\, u_j u_k\end{aligned}$$

since either $u_j = 0$ (Dirichlet) or $e_j (u_n) = 0$ (Neumann). The claim follows. □

Next we consider an elementary example. Let $M^n \subset \mathbb{R}^{n+1}$ be an embedded hypersurface and f a function defined in a neighborhood of M^n. We can compute the Hessian of f with respect to both the induced connection ∇ on M^n and the ambient flat connection D on $\mathbb{R}^{n+1}$. Comparing, we get

$$\nabla_i \nabla_j f - D_i D_j f = -\,\mathrm{II}_{ij} \frac{\partial f}{\partial \nu}$$

where ν is the unit normal to M^n used to define the second fundamental form II. In particular, if $M^n = S^n(1)$ is the unit sphere and if ℓ is a linear function on $\mathbb{R}^{n+1}$, then $\mathrm{II} = g$, $\frac{\partial \ell}{\partial \nu} = \ell$ and $DD\ell = 0$ so that

$$\nabla_i \nabla_j \ell = -g_{ij} \ell.$$

Tracing this, we see that

$$\Delta \ell = -n\ell$$

so that ℓ is an eigenfunction of the Laplacian on $S^n(1)$. In fact, the corresponding eigenvalue n is the first eigenvalue.

In the case where the Ricci curvature is bounded from below by a positive constant we have the following. In contrast to the pointwise gradient estimate of Li-Yau, Lichnerowicz' Theorem below uses an integral identity obtained by integration by parts (i.e., the divergence theorem). However, the estimates are related in that they both rely on computing the Laplacian of a gradient quantity.

THEOREM 1.134 (Lichnerowicz). *If (M^n, g) is a closed Riemannian manifold with $\mathrm{Rc} \geq K > 0$, then*

$$\lambda_1 \geq \frac{n}{n-1} K.$$

REMARK 1.135. *Note that this estimate is sharp on constant curvature spheres.*

PROOF. If $\Delta u + \lambda u = 0$, then

$$\begin{aligned}\frac{1}{2} \Delta |\nabla u|^2 &= |\nabla \nabla u|^2 + \langle \nabla \Delta u, \nabla u \rangle + \mathrm{Rc}(\nabla u, \nabla u) \\ &\geq \frac{1}{n} (\Delta u)^2 - \lambda |\nabla u|^2 + K |\nabla u|^2 \\ &= \frac{1}{n} \lambda^2 u^2 + (K - \lambda) |\nabla u|^2 .\end{aligned}$$

Integrating this, we have

$$0 \geq \frac{1}{n}\lambda^2 \int_M u^2 d\mu + (K - \lambda) \int_M |\nabla u|^2 \, d\mu.$$

Dividing by $\int_M u^2 d\mu$ and using the fact that $\lambda = \int_M |\nabla u|^2 \, d\mu \,/ \int_M u^2 d\mu$ implies

$$0 \geq \frac{1}{n}\lambda^2 + (K - \lambda)\,\lambda.$$

The theorem follows. □

To understand the case where M^n has boundary, we need an identity which follows from integration by parts.

LEMMA 1.136 (Reilly's Formula). *If (M^n, g) is a compact Riemannian manifold with boundary ∂M, then for any function f on M^n*

$$\int_M \left((\Delta f)^2 - |\nabla\nabla f|^2 \right) d\mu = \int_M \operatorname{Rc}(\nabla f, \nabla f)\, d\mu + \int_{\partial M} \left(\Delta_\partial f + H \frac{\partial f}{\partial \nu} \right) \frac{\partial f}{\partial \nu} d\sigma$$
$$+ \int_{\partial M} \left(- \left\langle \nabla_\partial f, \nabla_\partial \left(\frac{\partial f}{\partial \nu} \right) \right\rangle + \mathrm{II}\left(\nabla_\partial f, \nabla_\partial f\right) \right) d\sigma$$

where Δ_∂ and ∇_∂ denote the Laplacian and gradient on ∂M with its induced metric.

PROOF. We compute (summing on both i and j and setting $e_n = \nu$)

$$\begin{aligned}
\int_M (\Delta f)^2 \, d\mu &= \int_M f_{ii} f_{jj} d\mu = \int_M \left((f_{ii} f_j)_j - f_{iij} f_j \right) d\mu \\
&= \int_{\partial M} f_{ii} f_j \nu_j d\sigma - \int_M (f_{iji} - R_{jk} f_k)\, f_j d\mu \\
&= \int_{\partial M} (\Delta f)\, f_n d\sigma + \int_M \left((f_{ij})^2 + \operatorname{Rc}(\nabla f, \nabla f) \right) d\mu \\
&\quad - \int_{\partial M} f_{ij} f_j \nu_i d\sigma
\end{aligned}$$

so that

$$\begin{aligned}
\int_M \left((\Delta f)^2 - |\nabla\nabla f|^2 \right) d\mu &= \int_M \operatorname{Rc}(\nabla f, \nabla f)\, d\mu \\
&\quad + \int_{\partial M} \sum_{i \neq n} (f_{ii} f_n - f_{ni} f_i)\, d\sigma
\end{aligned}$$

since in the last term when $i = n$ we have a cancellation.

Now

$$\begin{aligned}\sum_{i\neq n} f_{ii} &= \sum_{i\neq n} \nabla_i\nabla_i f = \sum_{i\neq n}\left(\nabla_i\nabla_i f - \nabla_i^{\partial}\nabla_i^{\partial} f\right) + \Delta_{\partial} f\\ &= \sum_{i\neq n}\left(-\nabla_{e_i}e_i + \nabla^{\partial}_{e_i}e_i\right) f + \Delta_{\partial} f\\ &= H f_n + \Delta_{\partial} f\end{aligned}$$

where ∇^{∂} is the covariant derivative of ∂M. We also have

$$f_{ni} = \nabla_i\nabla_n f = e_i e_n f - \left(\nabla_{e_i} e_n\right) f = e_i\left(f_n\right) - \mathrm{II}_{ik}\, f_k$$

so that

$$\sum_{i\neq n} f_{ni} f_i = \left\langle \nabla_{\partial} f, \nabla_{\partial}\left(\frac{\partial f}{\partial \nu}\right)\right\rangle - \mathrm{II}_{ik}\, f_i f_k.$$

Therefore

$$\begin{aligned}\int_{\partial M}\sum_{i\neq n}\left(f_{ii}f_n - f_{ni}f_i\right) d\sigma &= \int_{\partial M}\left(H f_n + \Delta_{\partial} f\right) f_n d\sigma\\ &\quad + \int_{\partial M}\left(-\left\langle \nabla_{\partial} f, \nabla_{\partial}\left(\frac{\partial f}{\partial \nu}\right)\right\rangle + \mathrm{II}\left(\nabla_{\partial} f, \nabla_{\partial} f\right)\right) d\sigma\end{aligned}$$

and the lemma follows. □

The analogue of Lichnerowicz' Theorem for manifolds with boundary is the following.

THEOREM 1.137 (Reilly). *Let* (M^n, g) *be a compact manifold with boundary* ∂M *and* $\mathrm{Rc} \geq K > 0$.

(1) (Dirichlet problem) *If* $H(\partial M) \geq 0$, *then*

$$\lambda_D \geq \frac{n}{n-1}K.$$

(2) (Neumann problem) *If* $\mathrm{II}(\partial M) \geq 0$, *then*

$$\lambda_N \geq \frac{n}{n-1}K.$$

PROOF. Using Reilly's formula and

$$|\nabla\nabla u|^2 \geq \frac{1}{n}(\Delta u)^2 = \frac{1}{n}\lambda^2 u^2,$$

we compute

$$\begin{aligned}
\frac{n-1}{n}\lambda^2 \int_M u^2 d\mu &\geq \int_M \left((\Delta u)^2 - |\nabla\nabla u|^2 \right) d\mu \\
&= \int_M \mathrm{Rc}\,(\nabla u, \nabla u)\, d\mu + \int_{\partial M} \left(\Delta_\partial u + H \frac{\partial u}{\partial \nu} \right) \frac{\partial u}{\partial \nu} d\sigma \\
&+ \int_{\partial M} \left(-\left\langle \nabla_\partial u, \nabla_\partial \left(\frac{\partial u}{\partial \nu} \right) \right\rangle + \mathrm{II}\,(\nabla_\partial u, \nabla_\partial u) \right) d\sigma \\
&\geq \int_M K\, |\nabla u|^2\, d\mu + \int_{\partial M} \left(H \left(\frac{\partial u}{\partial \nu} \right)^2 + \mathrm{II}\,(\nabla_\partial u, \nabla_\partial u) \right) d\sigma
\end{aligned}$$

where we integrated by parts on ∂M. The theorem follows. □

Another beautiful application of Reilly's formula is the following eigenvalue estimate for minimal hypersurfaces.

THEOREM 1.138 (Choi-Wang). *Let (M^n, g) be a closed orientable manifold with $\mathrm{Rc} \geq K > 0$ and let $P^{n-1} \subset M^n$ be an embedded minimal hypersurface dividing M^n into two submanifolds M_1^n and M_2^n. Then*

$$\lambda_1 \left(P^{n-1} \right) \geq \frac{K}{2}.$$

PROOF. Taking into account orientations, we may assume $\partial M_1 = -\partial M_2 = P$. Suppose

$$\Delta_P u + \lambda u = 0$$

on P. Choose $i \in \{1, 2\}$ so that

$$\int_{\partial M_i} \mathrm{II}\,(\nabla_P u, \nabla_P u)\, d\sigma \geq 0.$$

We can do this since the sign of II depends on the orientation of ∂M_i. Define f on M_i so that

$$\Delta f = 0 \text{ on } M_i \quad \text{and} \quad f = u \text{ on } \partial M_i.$$

By Reilly's formula

$$\begin{aligned}
-\int_{M_i} |\nabla\nabla f|^2\, d\mu &= \int_{M_i} \mathrm{Rc}\,(\nabla f, \nabla f)\, d\mu + \int_{\partial M_i} (\Delta_P u) \frac{\partial f}{\partial \nu} d\sigma \\
&+ \int_{\partial M_i} \left(-\left\langle \nabla_P u, \nabla_P \left(\frac{\partial f}{\partial \nu} \right) \right\rangle + \mathrm{II}\,(\nabla_P u, \nabla_P u) \right) d\sigma.
\end{aligned}$$

Thus, by our curvature assumption on (M^n, g), we have

$$\begin{aligned}
0 &\geq K \int_{M_i} |\nabla f|^2\, d\mu - \lambda \int_{\partial M_i} u \frac{\partial f}{\partial \nu} d\sigma - \int_{\partial M_i} \left\langle \nabla_P u, \nabla_P \left(\frac{\partial f}{\partial \nu} \right) \right\rangle d\sigma \\
&= K \int_{M_i} |\nabla f|^2\, d\mu - 2\lambda \int_{\partial M_i} u \frac{\partial f}{\partial \nu} d\sigma.
\end{aligned}$$

However

$$2 \int_{\partial M_i} u \frac{\partial f}{\partial \nu} d\sigma = \int_{\partial M_i} \frac{\partial}{\partial \nu} \left(f^2 \right) d\sigma = \int_{M_i} \Delta \left(f^2 \right) d\mu = 2 \int_{M_i} |\nabla f|^2 \, d\mu$$

since $\Delta f = 0$. Therefore

$$0 \geq (K - 2\lambda) \int_{M_i} |\nabla f|^2 \, d\mu.$$

Since $\int_{M_i} |\nabla f|^2 \, d\mu > 0$, the theorem follows. □

11. The determinant of the Laplacian

11.1. The trace of the heat operator. Let (M^n, g) be a closed Riemannian manifold. We define the **heat operator**

$$e^{t\triangle} : L^2(M) \to L^2(M)$$

by

$$(e^{t\triangle} f)(y) \doteqdot \int_M h(x, y, t) f(x) d\mu(x),$$

for $f \in L^2(M)$, and where $h(\cdot, y, t)$ is the heat kernel. The heat operator is self-adjoint and takes a function f to the solution to the heat equation at time t with initial value f. (See Chapter VI of Chavel [**97**] for example.) The reason for the notation $e^{t\triangle}$ is that we may also express the heat operator formally as

$$e^{t\triangle} = \sum_{k=0}^{\infty} \frac{t^k \triangle^k}{k!},$$

where $\triangle^k = \underbrace{\triangle \circ \cdots \circ \triangle}_{k \text{ times}}$. Indeed, we formally compute that

$$\frac{\partial}{\partial t} e^{t\triangle} = \sum_{k=1}^{\infty} \frac{t^{k-1} \triangle^k}{(k-1)!} = \triangle \circ e^{t\triangle}$$

and

$$\lim_{t \to 0} e^{t\triangle} = \mathrm{id}.$$

Let $\{\varphi_k\}_{k=0}^{\infty}$ denote the eigenfunctions of the Laplacian $\triangle$ with corresponding eigenvalues $\{\lambda_k\}$ where

$$0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_k \leq \cdots$$

and $\lambda_k \to \infty$ as $k \to \infty$ from Weyl's asymptotic formula (1.138). Then $\{\varphi_k\}_{k=0}^{\infty}$ are also the eigenfunctions of the heat operator $e^{t\triangle}$, where the corresponding eigenvalues are $\left\{ e^{-t\lambda_k} \right\}$ where

$$1 = e^{-t\lambda_0} > e^{-t\lambda_1} \geq e^{-t\lambda_2} \geq \cdots \geq e^{-t\lambda_k} \geq \cdots.$$

In terms of these eigenfunctions and eigenvalues, we may write the heat kernel as

$$h(x,y,t) = \sum_{k=0}^{\infty} e^{-t\lambda_k}\varphi_k(x)\varphi_k(y).$$

The **trace of the heat operator** is defined by

$$\operatorname{tr}\left(e^{t\triangle}\right) \doteqdot \sum_{k=0}^{\infty} e^{-\lambda_k t}.$$

From (1.138): $\lambda_k \sim c_n \operatorname{Vol}(g)^{-2/n} k^{2/n}$, the series on the RHS converges for all $t > 0$.

EXERCISE 1.139. *To see that the series does indeed converge, show that for any $a > 1$ and $\varepsilon > 0$, the series $\sum_{k=0}^{\infty} a^{-k^{\varepsilon}}$ converges by making a comparison with a series to which one can apply the integral test.*

Since $\int_M \varphi_k(x)^2 d\mu(x) = 1$, we have

$$\operatorname{tr}\left(e^{t\triangle}\right) = \int_M h(x,x,t)d\mu(x).$$

The heat kernel along the diagonal has an asymptotic expansion of the form

$$h(x,x,t) = \frac{1}{(4\pi t)^{n/2}}\left\{\sum_{j=0}^{k} t^j \cdot c_j(x) + O(t^{k+1})\right\},$$

where the functions c_j depend only on the Riemann curvature tensor and its covariant derivatives. The first two terms of this expansion are given by

$$h(x,x,t) = \frac{1}{(4\pi t)^{n/2}}\left\{1 + \frac{R(x)}{6}t + O(t^2)\right\} \tag{1.140}$$

so that $c_0(x) \equiv 1$ and $c_1(x) = \frac{R(x)}{6}$. For more terms in this expansion the reader may refer to Gilkey's book [**228**]. Hence the trace of the heat operator has the expansion

$$\sum_{k=0}^{\infty} e^{-\lambda_k t} = \operatorname{tr}\left(e^{t\triangle}\right) = (4\pi t)^{-\frac{n}{2}}\left(\sum_{j=0}^{k} a_j \cdot t^j + O(t^{k+1})\right) \tag{1.141}$$

$$= \frac{1}{(4\pi t)^{n/2}}\left\{\operatorname{Vol}(M) + \frac{1}{6}\int_{M^n} R(x)d\mu(x)\cdot t + O(t^2)\right\} \tag{1.142}$$

as $t \to 0$, where $\{a_j\}_{j=0}^{\infty}$ are constants. In particular, $a_0 = \operatorname{Vol}(M)$ and $a_1 = \frac{1}{6}\int_{M^n} R(x)d\mu(x)$.

EXERCISE 1.140. *Show that* $\operatorname{tr}(e^{t\triangle}) - 1 = \sum_{k=1}^{\infty} e^{-t\lambda_k}$ *is an exponentially decreasing function of t as $t \to \infty$.*

Let $N(\lambda)$ denote the number of eigenvalues less than equal to λ counting multiplicity. Equation (1.142) implies, using the Karamata theorem (see p. 446 of [**208**]),

$$N(\lambda) \sim \frac{\omega_n \operatorname{Vol}(g)}{(2\pi)^n} \lambda^{n/2} \tag{1.143}$$

as $\lambda \to \infty$. This implies the Weyl asymptotic formula (1.138) by taking $\lambda = \lambda_k$:

$$k = N(\lambda_k) \sim \frac{\omega_n \operatorname{Vol}(g)}{(2\pi)^n} (\lambda_k)^{n/2}$$

as $k \to \infty$. (See also [**97**], p. 155.)

11.2. The zeta function and its regularization. Recall that the **Riemann zeta function**

$$\zeta_{\mathrm{Rm}} : \{s \in \mathbb{C} : \operatorname{Re}(s) > 1\} \to \mathbb{C}$$

is defined by

$$\zeta_{\mathrm{Rm}}(s) \doteqdot \sum_{k=1}^{\infty} k^{-s}.$$

Since

$$\sum_{k=1}^{\infty} \left| k^{-s} \right| = \sum_{k=1}^{\infty} k^{-\operatorname{Re}(s)},$$

the series on the RHS converges absolutely if and only if $\operatorname{Re}(s) > 1$. It is well-known that one can extend ζ_{Rm} meromorphically to all of $\mathbb{C}$. We shall prove this in more generality later.

Again let (M^n, g) be a closed Riemannian manifold. The Laplacian $-\Delta$ acting on functions is a nonnegative operator and has a discrete spectrum. We define the **zeta function** using the eigenvalues of the Laplacian instead of the positive integers by

$$\zeta(s) \doteqdot \sum_{k=1}^{\infty} \lambda_k^{-s}.$$

By Weyl's asymptotic formula, we see that the series on the RHS converges absolutely if and only if $\operatorname{Re}(s) > \frac{n}{2}$. Note that when the Riemannian manifold (M^n, g) is the unit circle S^1, the zeta function ζ is, up to a constant, the Riemann zeta function ζ_{Rm}.

To meromorphically continue the zeta function $\zeta(s)$ we recall the important relationship between the heat kernel $e^{t\Delta}$, the **Gamma function** $\Gamma(s)$, and the zeta function $\zeta(s)$. First recall that the Gamma function is given by

$\Gamma(s) \doteqdot \int_0^\infty t^s e^{-t} \frac{dt}{t}$. We compute

$$\begin{aligned}
\zeta(s) \cdot \Gamma(s) &= \sum_{k=1}^{\infty} \lambda_k^{-s} \cdot \int_0^\infty t^s e^{-t} \frac{dt}{t} \\
&= \sum_{k=1}^{\infty} \int_0^\infty \left(\frac{t}{\lambda_k} \right)^s e^{-\frac{t}{\lambda_k} \lambda_k} \frac{d\left(t/\lambda_k\right)}{t/\lambda_k} \\
&= \int_0^\infty \left(\sum_{k=1}^{\infty} e^{-\tau \lambda_k} \right) \tau^s \frac{d\tau}{\tau},
\end{aligned}$$

where to obtain the last line we made the substitution $\tau = t/\lambda_k$. Since

$$\operatorname{tr}(e^{t\triangle}) = \sum_{k=0}^{\infty} e^{-t\lambda_k},$$

we have the formula

$$\zeta(s) = \frac{1}{\Gamma(s)} \int_0^\infty \left(\operatorname{tr}(e^{t\triangle}) - 1 \right) t^s \frac{dt}{t}.$$

REMARK 1.141. *The* ***Mellin transform*** *of a function f is defined by*

$$\left(\mathrm{M}\, f\right)(s) \doteqdot \int_0^\infty f\left(t\right) t^s \frac{dt}{t}.$$

So up to the Gamma function, the zeta function is the Mellin transform of the trace of the heat kernel.

We now write the zeta function as

$$\zeta(s) = \frac{1}{\Gamma(s)} \int_0^1 \left(\operatorname{tr}(e^{t\triangle}) - 1 \right) t^s \frac{dt}{t} + \frac{\phi(s)}{\Gamma(s)},$$

where

$$\phi(s) \doteqdot \int_1^\infty \left(\operatorname{tr}(e^{t\triangle}) - 1 \right) t^s \frac{dt}{t}$$

is an entire function (this is because $\operatorname{tr}(e^{t\triangle}) - 1$ is exponentially decreasing as $t \to \infty$.) Hence we have from (1.141)

$$\zeta(s) = \frac{1}{\Gamma(s)} \int_0^1 (4\pi t)^{-\frac{n}{2}} \left(\sum_{j=0}^{k} a_j t^j + O(t^{k+1}) \right) t^s \frac{dt}{t} + \frac{\phi(s)}{\Gamma(s)}. \tag{1.144}$$

We compute that

$$\int_0^1 t^{j-\frac{n}{2}+s} \frac{dt}{t} = \frac{1}{j - \frac{n}{2} + s}\, t^{j-\frac{n}{2}+s} \Big|_0^1 = \frac{1}{j - \frac{n}{2} + s} \text{ for } \operatorname{Re}(s) > \frac{n}{2} - j. \tag{1.145}$$

Substituting (1.145) into (1.144), we obtain

$$\zeta(s) = \frac{(4\pi)^{-n/2}}{\Gamma(s)} \left(\sum_{j=0}^{k} \frac{a_j}{j - \frac{n}{2} + s} \right) + \alpha_k(s),$$

where $\alpha_k(s)$ is a meromorphic function on $\operatorname{Re}(s) > \frac{n}{2} - k - 1$. Since $\frac{1}{\Gamma(s)}$ is a meromorphic function and $k \in \mathbb{N}$ is arbitrary, by taking $k \to \infty$ we have meromorphically extended $\zeta(s)$ to all of $\mathbb{C}$.

11.3. The determinant of the Laplacian. Formally, the determinant of the Laplacian is given by

$$\det \Delta = \prod_{k=1}^{\infty} \lambda_k .$$

Of course, since $\lim_{k\to\infty} \lambda_k = \infty$, the product on the RHS diverges. However, we can use the zeta function to define the determinant of the Laplacian as follows. We compute

$$\zeta'(s) = -\sum_{k=1}^{\infty} \log \lambda_k \cdot \lambda_k^{-s},$$

for $s \in \mathbb{C}$ such that $\operatorname{Re}(s) > \frac{n}{2}$. Formally, we have

$$\zeta'(0) = -\sum_{k=1}^{\infty} \log \lambda_k,$$

which implies that formally,

$$e^{-\zeta'(0)} = \prod_{k=1}^{\infty} \lambda_k .$$

Now $\zeta(s)$ is defined for all $s \in \mathbb{C}$ by meromorphic extension. We shall show that $\zeta(s)$ is regular at $s = 0$. The **determinant of the Laplacian** is then defined by

$$\det \Delta \doteqdot e^{-\zeta'(0)}.$$

In particular, $\zeta'(0) = -\log \det \Delta$.

We now consider the determinant of the Laplacian on a closed Riemannian surface. In this case, by (1.142) and the Gauss-Bonnet formula

$$\operatorname{tr}(e^{t\triangle}) = \frac{1}{4\pi t}\left(\operatorname{Area}(M) + \frac{2\pi}{3}\chi(M)\cdot t + O(t^2)\right).$$

Although it is not possible to compute the determinant of the Laplacian of a given metric, one can compute the difference of the determinant of the Laplacian of two conformally equivalent metrics on a surface.

We first compute the conformal variation of the determinant of the Laplacian. Let g be a metric on a surface and $g(u)$ a family of metrics with

$$g(0) = g \text{ and } \frac{\partial}{\partial u} g(0) = \psi \cdot g,$$

where ψ is a C^∞ function on M. The variation of the Laplacian is then given by (2.21):

$$\left.\frac{d}{du}\right|_{u=0} \triangle_{g(u)} = -\psi \triangle_g .$$

We compute that the variation of the zeta function is

$$\begin{aligned}
\frac{d}{du}\Big|_{u=0}\zeta_{g(u)}(s) &= \frac{d}{du}\Big|_{u=0}\left\{\frac{1}{\Gamma(s)}\int_0^\infty\left(\operatorname{tr}(e^{t\triangle_{g(u)}})-1\right)t^s\frac{dt}{t}\right\}\\
&= \frac{1}{\Gamma(s)}\int_0^\infty \frac{d}{du}\Big|_{u=0}\operatorname{tr}(e^{t\triangle_{g(u)}})\cdot t^{s-1}dt\\
&= \frac{1}{\Gamma(s)}\int_0^\infty \operatorname{tr}\left(t\,\frac{d}{du}\Big|_{u=0}\triangle_{g(u)}\circ e^{t\triangle_g}\right)\cdot t^{s-1}dt\\
&= \frac{1}{\Gamma(s)}\int_0^\infty \operatorname{tr}\left(-\psi\cdot\triangle_g\circ e^{t\triangle_g}\right)\cdot t^{s}dt\\
&= \frac{1}{\Gamma(s)}\int_0^\infty \operatorname{tr}\left(-\psi\cdot\frac{\partial}{\partial t}\left(e^{t\triangle_g}-\frac{1}{\mathrm{A}}\right)\right)\cdot t^{s}dt\\
&= -\frac{1}{\Gamma(s)}\int_0^\infty \frac{\partial}{\partial t}\left[\operatorname{tr}\left(\psi\left(e^{t\triangle_g}-\frac{1}{\mathrm{A}}\right)\right)\right]\cdot t^{s}dt\\
&= \frac{s}{\Gamma(s)}\int_0^\infty \operatorname{tr}\left(\psi\left(e^{t\triangle_g}-\frac{1}{\mathrm{A}}\right)\right)\cdot t^{s-1}dt
\end{aligned}$$

where $\frac{1}{\mathrm{A}}$ denotes $f\mapsto\frac{1}{\mathrm{A}}\int_M f\,d\mu$ and $\mathrm{A}=\operatorname{Area}(g)$. Thus we need to compute

$$\begin{aligned}
\operatorname{tr}\left(\psi\cdot e^{t\triangle_g}\right) &= \sum_{k=0}^\infty\left\langle\psi\cdot e^{t\triangle_g}(\varphi_k),\varphi_k\right\rangle_{L^2}\\
&= \sum_{k=0}^\infty\int_M\psi(y)\int_M h(x,y,t)\varphi_k(x)dA(x)\cdot\varphi_k(y)dA(y)\\
&= \sum_{k=0}^\infty\int_M\psi(y)\cdot e^{-t\lambda_k}\varphi_k(y)\cdot\varphi_k(y)dA(y)\\
&= \int_M\psi(y)\cdot h(y,y,t)dA(y).
\end{aligned}$$

Hence the variation of the zeta function is given by

$$\frac{d}{du}\Big|_{u=0}\zeta_{g(u)}(s)=\frac{s}{\Gamma(s)}\int_0^\infty\left(\int_M\psi(x)\left(h(x,x,t)-\frac{1}{\mathrm{A}}\right)dA(x)\right)t^{s-1}dt.$$

The integrand has an expansion of the form

$$\int_M\psi(x)\left(h(x,x,t)-\frac{1}{\mathrm{A}}\right)dA(x)=\sum_{k=-1}^{k}d_jt^j+O(t^{k+1}),\tag{1.146}$$

where d_j are real constants.

The variation of the logarithm of the determinant of the Laplacian is given by

$$\left.\frac{d}{du}\right|_{u=0} \log\det\triangle_{g(u)} = -\left.\frac{d}{du}\right|_{u=0} \zeta'_{g(u)}(0)$$
$$= -\left.\frac{d}{ds}\right|_{s=0}\left[\frac{s}{\Gamma(s)}\int_0^\infty\left(\int_M \psi(x)\left(h(x,x,t)-\frac{1}{\mathrm{A}}\right)dA(x)\right)t^{s-1}dt\right].$$

Since $\Gamma(s) = \frac{1}{s} + O\left(1\right)$, we have

$$\left.\frac{d}{du}\right|_{u=0} \log\det\triangle_{g(u)}$$
$$= -\left.\frac{d}{ds}\right|_{s=0}\left[\left(s^2+O\left(s^3\right)\right)\int_0^\infty\left(\sum_{j=-1}^{k} d_j\cdot t^j + O(t^{k+1})\right)\cdot t^{s-1}dt\right]$$
$$= -\left.\frac{d}{ds}\right|_{s=0}\left[s^2\left(\sum_{j=-1}^{k}\frac{d_j}{s+j}+\beta(s)\right)\right],$$

where $\beta(s)$ is analytic on $\{s\in\mathbb{C} : \mathrm{Re}(s) > -k-1\}$. Hence

$$\left.\frac{d}{du}\right|_{u=0} \log\det\triangle_{g(u)} = -d_0.$$

Substituting (1.140) with $n=2$ into (1.146), we obtain

$$d_0 = \int_M \psi(x)\left(\frac{R(x)}{24\pi} - \frac{1}{\mathrm{Area}\,(g)}\right)dA(x).$$

Thus

LEMMA 1.142. *The variation in a conformal class of the determinant of the Laplacian on a Riemannian surface is given by if*

$$g(0) = g \quad \text{and} \quad \frac{\partial}{\partial u}g(0) = \psi\cdot g,$$

then

$$\left.\frac{d}{du}\right|_{u=0} \log\det\triangle_{g(u)} = -\int_M \psi(x)\left(\frac{R_g(x)}{24\pi} - \frac{1}{\mathrm{Area}(M,g)}\right)dA_g(x).$$

Now let h and g be two metrics on M^2 conformally related by

$$g = e^{\psi}h$$

where $\psi : M^2 \to \mathbb{R}$. Define

$$g(u) \doteqdot e^{u\psi}h$$

for $u\in[0,1]$. We have

$$\frac{\partial}{\partial u}g(u) = \psi\cdot g(u).$$

The difference of the logarithms of the determinants of the Laplacians of g and h is:

$$\begin{aligned}
&\log\det\triangle_g - \log\det\triangle_h \\
&= \int_0^1 \frac{\partial}{\partial u}\log\det\triangle_{g(u)} du \\
&= -\int_0^1\int_M \psi(x)\left(\frac{R_{g(u)}(x)}{24\pi} - \frac{1}{\operatorname{Area}(M^2, g(u))}\right) dA_{g(u)}(x)\, du.
\end{aligned}$$

Using the formulas

$$R_{g(u)} = e^{-u\psi}\left(-u\triangle_h\psi + R_h\right)$$

and

$$\frac{\partial}{\partial u}\log\left(\operatorname{Area}\left(M^2, g(u)\right)\right) = \frac{1}{\operatorname{Area}(M^2, g(u))}\int_M \psi(x) dA_{g(u)}(x)\, du,$$

we compute that

$$\begin{aligned}
\log\det\triangle_g - \log\det\triangle_h &= -\frac{1}{24\pi}\int_0^1\int_M \psi\left(-u\triangle_h\psi + R_h\right) dA_h\, du \\
&\quad + \int_0^1 \frac{\partial}{\partial u}\log\left(\operatorname{Area}\left(M^2, g(u)\right)\right) du \\
&= -\frac{1}{24\pi}\int_M\left(\frac{1}{2}|\nabla\psi|_h^2 + \psi R_h\right) dA_h \qquad (1.147)\\
&\quad + \log\left(\operatorname{Area}\left(M^2, g\right)\right) - \log\left(\operatorname{Area}\left(M^2, h\right)\right),
\end{aligned}$$

where we integrated by parts in space to obtain the last line. Hence we have (see also [**433**], [**428**], [**412**]):

PROPOSITION 1.143. *If h and g are two metrics on M^2 with the same area and conformally related by $g = e^{\psi}h$, then the difference of the determinants of the Laplacian is*

$$\log\det\triangle_g - \log\det\triangle_h = -\frac{1}{48\pi}\int_M\left(|\nabla\psi|_h^2 + 2R_h\psi\right) dA_h,$$

*where R_h is the scalar curvature of h. In particular, this difference is the same as the relative energy defined by (**??**).*

The determinant of the Laplacian on a surface is the energy functional for the Ricci flow; see Volume 2.

12. Monotonicity for harmonic functions and maps

For comparison with other monotonicity formulas in this book, we recall a fundamental monotonicity formula used in the study of harmonic functions and maps.

LEMMA 1.144. *If u is a harmonic function in a ball $B(p,r) \subset \mathbb{R}^n$, then*

$$(n-2)\int_{B(p,r)} |\nabla u|^2 \, d\mu = r\int_{\partial B(p,r)} \left(|\nabla u|^2 - 2\left(\frac{\partial u}{\partial r}\right)^2 \right) d\sigma. \tag{1.148}$$

PROOF. We derive the more general identity

$$\int_{B(p,r)} \left(\delta_{ij}|\nabla u|^2 - 2\nabla_i u \nabla_j u\right) \nabla_i V_j d\mu = \int_{\partial B(p,r)} \left(\delta_{ij}|\nabla u|^2 - 2\nabla_i u \nabla_j u\right) \nu_i V_j d\sigma \tag{1.149}$$

for any $V \in C^\infty(B(p,r))$ and where $\nu = \frac{\partial}{\partial r}$ is the unit outward normal. Taking $V(x) = x - p = r\frac{\partial}{\partial r}$ in (1.149), we have $\nabla_i V_j = \delta_{ij}$ and therefore obtain (1.148). Now since $\Delta u = 0$, two integrations by parts yield

$$\begin{aligned} 2\int_{\partial B(p,r)} (\nabla u \cdot V)\cdot(\nabla u \cdot \nu)\, d\sigma &= 2\int_{B(p,r)} \nabla(\nabla u \cdot V)\cdot \nabla u \; d\mu \\ &= \int_{B(p,r)} \left(\nabla_j |\nabla u|^2 V_j + 2\nabla_i V_j \nabla_i u \nabla_j u\right) \, d\mu \\ &= \int_{B(p,r)} \left(-|\nabla u|^2 \nabla_j V_j + 2\nabla_i V_j \nabla_i u \nabla_j u\right) \, d\mu \\ &\quad + \int_{\partial B(p,r)} |\nabla u|^2 \nu_j V_j \; d\sigma, \end{aligned}$$

which is (1.149). A more geometric way to derive (1.149) is to consider the 1-parameter family of functions

$$u_s(x) = u(x + sV(x)),$$

where $V \in C_0^\infty(B(p,r))$, which satisfy

$$\left.\frac{d}{ds}\right|_{s=0} u_s(x) = \nabla u \cdot V.$$

Since u is a stationary point of the energy $E(u) \doteqdot \int_{B(p,r)} |\nabla u|^2 d\mu$ under variations with $u_s = u$ on $\partial B(p,r)$, we have

$$0 = \left.\frac{d}{ds}\right|_{s=0} E(u_s) = \int_{B(p,r)} \left(-|\nabla u|^2 \nabla_j V_j + 2\nabla_i V_j \nabla_i u \nabla_j u\right) \, d\mu$$

for $V \in C_0^\infty(B(p,r))$. From this one can derive (1.149). □

A direct consequence of Lemma 1.144 is the following monotonicity formula.

THEOREM 1.145. *If u is a harmonic in $B(p,r) \subset \mathbb{R}^n$, then*

$$\frac{\partial}{\partial r}\left(r^{2-n}\int_{B(p,r)} |\nabla u|^2 \, d\mu\right) = 2r^{2-n}\int_{\partial B(p,r)} \left(\frac{\partial u}{\partial r}\right)^2 d\sigma \geq 0$$

and

$$R^{2-n}\int_{B(p,R)}|\nabla u|^2\,d\mu - r^{2-n}\int_{B(p,r)}|\nabla u|^2\,d\mu = 2\int_{B(p,R)-B(p,r)}|x|^{2-n}\left(\frac{\partial u}{\partial r}\right)^2 d\mu.$$

PROOF. This follows from

$$\frac{\partial}{\partial r}\left(r^{2-n}\int_{B(p,r)}|\nabla u|^2\,d\mu\right) = r^{2-n}\int_{\partial B(p,r)}|\nabla u|^2\,d\sigma + (2-n)\,r^{1-n}\int_{B(p,r)}|\nabla u|^2\,d\mu$$

and the lemma. □

See §2.8 for more about harmonic maps and their relation to DeTurck's trick (where one couples the Ricci flow with the harmonic map flow to produce a strictly parabolic system in order to prove short time existence for the Ricci flow). Harmonic maps also provide good parametrizations. There are a number of sources for the study of harmonic maps, including [**195**], [**254**], [**277**], [**350**], and [**448**].

13. Lie groups and left invariant metrics

Recall that a **Lie group** is a C^∞ manifold G with the structure of a group, such that the map $\mu : G \times G \to G$, defined by $\mu(\sigma,\tau) = \sigma\cdot\tau^{-1}$ is C^∞. Given $\sigma \in G$, we define left multiplication by σ, $\sigma_L : G \to G$ where $\sigma_L(\tau) = \sigma\cdot\tau$ and right multiplication by σ, $\sigma_R : G \to G$ where $\sigma_R(\tau) = \tau\cdot\sigma$.

DEFINITION 1.146. *A Riemannian metric g on G is **left-invariant** if for any $\sigma \in G$, σ_L is an isometry of (G,g): $(\sigma_L)^* g = g$.*

The connection and curvature of a left-invariant metric may be computed algebraically (and metrically) using the following.

LEMMA 1.147. *Let g be a left-invariant metric on G. If X, Y, Z, W are left-invariant vector fields, then*

(1) *the Levi-Civita connection is given by:*

$$\langle \nabla_X Y, Z\rangle = \frac{1}{2}\left(\langle [X,Y], Z\rangle - \langle [X,Z], Y\rangle - \langle [Y,Z], X\rangle\right), \tag{1.150}$$

(2) *the Riemann curvature tensor is given by:*

$$\langle \operatorname{Rm}(X,Y)Z, W\rangle = \langle \nabla_X Z, \nabla_Y W\rangle - \langle \nabla_Y Z, \nabla_X W\rangle - \left\langle \nabla_{[X,Y]} Z, W\right\rangle, \tag{1.151}$$

so that in particular

$$\langle \operatorname{Rm}(X,Y)Y, X\rangle = \langle \nabla_X Y, \nabla_Y X\rangle - \langle \nabla_Y Y, \nabla_X X\rangle - \left\langle \nabla_{[X,Y]} Y, X\right\rangle. \tag{1.152}$$

PROOF. 1). The formula for the Levi-Civita connection follows from (1.3) and the fact that $X\langle Y,Z\rangle = 0$, etc.

2). By the definition of the Riemann curvature tensor

$$\begin{aligned}\langle \mathrm{Rm}(X,Y)Z,W\rangle &= \langle \nabla_X\nabla_Y Z, W\rangle - \langle \nabla_Y\nabla_X Z, W\rangle - \left\langle \nabla_{[X,Y]}Z, W\right\rangle \\ &= X\langle \nabla_Y Z, W\rangle - \langle \nabla_Y Z, \nabla_X W\rangle \\ &\quad - Y\langle \nabla_X Z, W\rangle + \langle \nabla_X Z, \nabla_Y W\rangle - \left\langle \nabla_{[X,Y]}Z, W\right\rangle .\end{aligned}$$

The formula for the Riemann curvature tensor now follows from this and the fact that $\langle \nabla_X Z, W\rangle$ is a constant function. □

On a general manifold, the covariant derivative with respect to Killing vector fields can be expressed in terms of the Lie brackets.

LEMMA 1.148. *If (M,g) is a Riemannian manifold and X,Y, and Y are Killing vector fields, then*

$$\langle \nabla_X Y, Z\rangle = \frac{1}{2}\left(\langle [X,Y], Z\rangle + \langle [X,Z], Y\rangle + \langle [Y,Z], X\rangle\right). \tag{1.153}$$

PROOF. Since Y is a Killing vector field, we have by (1.21)

$$0 = \langle \nabla_X Y, Z\rangle + \langle \nabla_Z Y, X\rangle .$$

We have two more similar formulas following from the fact that X and Z are Killing vector fields. These three formulas imply:

$$\begin{aligned}\langle \nabla_X Y, Z\rangle = -\langle \nabla_Z Y, X\rangle &= \langle [Y,Z], X\rangle - \langle \nabla_Y Z, X\rangle = \langle [Y,Z], X\rangle + \langle \nabla_X Z, Y\rangle \\ &= \langle [Y,Z], X\rangle + \langle [X,Z], Y\rangle + \langle \nabla_Z X, Y\rangle \\ &= \langle [Y,Z], X\rangle + \langle [X,Z], Y\rangle - \langle \nabla_Y X, Z\rangle \\ &= \langle [Y,Z], X\rangle + \langle [X,Z], Y\rangle + \langle [X,Y], Z\rangle - \langle \nabla_X Y, Z\rangle ,\end{aligned}$$

and the lemma follows. □

A left-invariant metric is **bi-invariant** if it is also invariant under right multiplication.

EXERCISE 1.149. *Show that if (G,g) is bi-invariant and if X is a left-invariant vector field, then X is a Killing vector field.*

LEMMA 1.150. *Let g be a bi-invariant metric on G. If X,Y,Z,W are left-invariant vector fields, then*

(1) *the Levi-Civita connection is given by:*

$$\nabla_X Y = \frac{1}{2}[X,Y],$$

(2) *the Riemann curvature tensor is given by:*

$$\langle \mathrm{Rm}(X,Y)Z,W\rangle = \frac{1}{4}\left(\langle [X,W],[Y,Z]\rangle - \langle [X,Z],[Y,W]\rangle\right). \tag{1.154}$$

PROOF. 1) Since in this case, X,Y and Z are Killing vector fields, we have both (1.150) and (1.153). Adding these two equations together implies:

$$2\langle \nabla_X Y, Z\rangle = \langle [X,Y], Z\rangle .$$

□

2) From (1.151) and part 1), we have

(1.155)

$$\langle \mathrm{Rm}(X,Y)Z,W\rangle = \frac{1}{4}\langle [X,Z],[Y,W]\rangle - \frac{1}{4}\langle [Y,Z],[X,W]\rangle - \frac{1}{2}\langle [[X,Y],Z],W\rangle .$$

On the other hand, by the Jacobi identity

$$\begin{aligned}\langle [[X,Y],Z],W\rangle &= -\langle [[Y,Z],X],W\rangle - \langle [[Z,X],Y],W\rangle \\ &= 2\langle \nabla_X [Y,Z],W\rangle + 2\langle \nabla_Y [Z,X],W\rangle \\ &= -2\langle [Y,Z],\nabla_X W\rangle - 2\langle [Z,X],\nabla_Y W\rangle \\ &= -\langle [Y,Z],[X,W]\rangle - \langle [Z,X],[Y,W]\rangle ,\end{aligned}$$

and the result follows from substituting this into (1.155).

Corollary 1.151. *A bi-invariant metric on a Lie group G has nonnegative sectional curvature.*

Proof. From (1.154) we have

$$\langle \mathrm{Rm}(X,Y)Y,X\rangle = -\frac{1}{4}\langle [X,Y],[Y,X]\rangle = \frac{1}{4}|[X,Y]|^2 \geq 0.$$

□

Left-invariant metrics on Lie groups (and more generally, homogeneous metrics) provide nice examples to study the behavior of the Ricci flow. See §4.10.

14. Bieberbach Theorem

In this section we present the Bieberbach Theorem classifying closed flat manifolds. The ideas of the proof are based on Gromov's proof of his almost flat manifolds theorem [**241**], [**66**]. The main reference we follow for this section is Buser's expository paper [**65**]. First we give some definitions.

Definition 1.152. *Let G be a Hausdorff topological group.*

(1) *A **discrete subgroup** of G is a subgroup which is a discrete subset, that is, each element g of the subset is contained in a neighborhood whose intersection with the subset is g.*
(2) *A closed subgroup $H \subset G$ is **uniform** if the left coset space $G/H \doteqdot \{gH : g \in G\}$ is a compact space.*

Definition 1.153. *An n-**dimensional crystallographic group** is a discrete, uniform group of isometries (rigid motions) of $\mathbb{R}^n$.*

If G is an n-dimensional crystallographic group, then $\mathbb{R}^n/G$ with the quotient metric induced by the euclidean metric is a closed flat Riemannian manifold. Conversely, a closed flat Riemannian n-manifold is the quotient of $\mathbb{R}^n$ by an n-dimensional crystallographic group. The main result on the classification of flat manifolds is the following.

THEOREM 1.154 (Bieberbach). *If G is an n-dimensional crystallographic group, then G contains a subgroup generated by n linearly independent translations. This subgroup is normal and of finite index.*

In particular, if M^n is a closed flat Riemannian n-manifold, then a finite cover of M^n is a flat torus, which is the quotient of $\mathbb{R}^n$ by n linearly independent translations. The rest of this section is devoted to a sketch of the proof of the above theorem.

Let $\mathrm{AO}(n)$ denote the group of isometries of $\mathbb{R}^n$. An isometry may be written uniquely in the form

$$\alpha(x) = Ax + a$$

where $A \doteqdot \mathrm{rot}(\alpha) \in \mathrm{O}(n)$ is called the **rotational part** and $a \doteqdot \mathrm{trans}(\alpha) \in \mathbb{R}$ is called the **translational part**. Let $[\alpha, \beta] = \alpha\beta\alpha^{-1}\beta^{-1}$ denote the **bracket** of α and β. One easily computes that the rotational and translational parts of the bracket are give by

$$\mathrm{rot}([\alpha, \beta]) = [A, B]$$
$$\mathrm{trans}([\alpha, \beta]) = (A - I)\, b + (I - [A, B])\, b + A(I - B)\, A^{-1}a.$$

We also define the **distance function on the orthogonal group** $\mathrm{O}(n)$ by

$$d(A, B) \doteqdot \sup_{x \in \mathbb{R}^n - \{0\}} \frac{\left|AB^{-1}x - x\right|}{|x|} = \max_{x \in S^{n-1}} \left|AB^{-1}x - x\right|.$$

EXERCISE 1.155. *Show that $d(A, B) = d(B, A)$.*

We define the **norm** of an orthogonal matrix by

$$\|A\| \doteqdot d(A, I) = \max_{x \in S^{n-1}} |Ax - x|,$$

which is the operator norm of $A - I$.

The following result implies that the group commutator of two orthogonal matrices which are close to the identity is much closer to the identity than the two matrices.

LEMMA 1.156 (Estimate for the norm of the commutator). *If $A, B \in \mathrm{O}(n)$, then*

$$\boxed{\|[A, B]\| \leq 2\, \|A\| \cdot \|B\|.}$$

In particular, if $\|A\|, \|B\| \leq \varepsilon$, then $\|[A, B]\| \leq 2\varepsilon^2$.

REMARK 1.157. *On the other hand, if $\|A\| \leq 1/2$, then $\|[A, B]\| \leq \|B\|$. Thus, taking a commutator with an element of norm less than $1/2$ decreases the norm.*

PROOF. We have the identity

$$[A, B] - I = \{(A - I)(B - I) - (B - I)(A - I)\}\, A^{-1}B^{-1}.$$

Thus

$$\begin{aligned}\|[A,B]\| &= \max_{x\in S^{n-1}} \left|\{(A-I)(B-I)-(B-I)(A-I)\}A^{-1}B^{-1}x\right| \\ &= \max_{x\in S^{n-1}} \left|\{(A-I)(B-I)-(B-I)(A-I)\}x\right| \\ &\leq 2\|A\|\cdot\|B\|.\end{aligned}$$

□

The follow establishes the existence of almost translations in almost any direction in a crystallographic group.

Lemma 1.158 (Existence of almost translations in almost any direction). *Let G be an n-dimensional crystallographic group. For every $v \in S^{n-1}$ and $\varepsilon, \delta > 0$, there exists an element $\gamma \in G$, where $\gamma(x) = Cx + c$, such that*

1. *the translational part points almost in the direction of v: $c \neq 0$ and $\measuredangle(c, v) < \delta$*
2. *the rotational part is small: $\|C\| < \varepsilon$.*

The following is the main lemma used to prove the existence of n linearly independent translations. Roughly speaking, it says that an almost translation is a pure translation. So it goes well with the previous lemma.

Lemma 1.159 (Elements with small rotational parts are translations). *Let G be an n-dimensional crystallographic group. If $\alpha \in G$, where $\alpha(x) = Ax + a$, is such that $\|A\| \leq 1/2$, then $A = I$; that is, α is a pure translation.*

The Bieberbach Theorem is a consequence of the following.

Proposition 1.160. *If G is an n-dimensional crystallographic group, then G contains a subgroup generated by n linearly independent translations.*

Proof of the proposition from the lemma. Choose any linearly independent set of n vectors $v_1, \ldots, v_n$ in $\mathbb{R}^n$ such as the standard basis. Since the vectors are linearly independent, there exists $\delta > 0$ such that

$$\measuredangle(v_i, v_j) > 2\delta \quad \text{for all } i \neq j. \tag{1.156}$$

By Lemma 1.158, taking $\varepsilon = 1/2$, δ as above and $v = v_i/|v_i|$, there exist n elements $\gamma_1, \ldots, \gamma_n \in G$, where $\gamma_i(x) = C_i x + c_i$, such that $c_i \neq 0$,

$$\measuredangle(c_i, v_i) < \delta, \tag{1.157}$$

and $\|C_i\| < 1/2$. Inequalities (1.156) and (1.157) imply that $c_1, \ldots, c_n$ are linearly independent. Furthermore, Lemma 1.159 implies $C_i = I$ for all i. Hence $\gamma_1, \ldots, \gamma_n \in G$ are linearly independent pure translations. □

Now we give in a number of steps the

Proof of Lemma 1.159. Step 1. *Among the elements of G with nonzero small rotational part, choose one, which we call α, with the smallest translational part.* In particular, let

$$S \doteqdot \{\gamma \in G : 0 < \|C\| \leq 1/2\}.$$

If the lemma is not true, then the set is nonempty. Since G is discrete and $\mathrm{O}(n)$ is compact, there exists an element $\alpha \in S$ such that

$$|a| = |\mathrm{trans}(\alpha)| = \min_{\gamma \in S} |\mathrm{trans}(\gamma)|.$$

STEP 2. $A = \mathrm{rot}(\alpha)$ *yields an orthogonal decomposition of* $\mathbb{R}^n$ *into the subspace of maximal rotation directions and its orthogonal complement.* In particular, let $\alpha \in G$ be an element as given by Step 1 and define

$$E_A \doteqdot \mathrm{span}\{x \in \mathbb{R}^n : |Ax - x| = \|A\| \cdot |x|\}.$$

E_A is a nonempty and we claim that it is also an A-invariant subspace. Indeed, if $x \in E_A$, then

$$|A(Ax) - Ax| = |A(Ax - x)| = |Ax - x| = \|A\| \cdot |x| = \|A\| \cdot |Ax|.$$

Clearly, if $x \in E_A$ and $c \in \mathbb{R}$, then $cx \in E_A$. Since A is orthogonal,

$$(E_A)^\perp \doteqdot \{x \in \mathbb{R}^n : \langle x, y \rangle = 0 \text{ for all } y \in E_A\}$$

is also A-invariant. We define the norm of A restricted to $(E_A)^\perp$ by

$$\|A\|^\perp \doteqdot \max_{x \in (E_A)^\perp - \{0\}} \frac{|Ax - x|}{|x|}$$

if $(E_A)^\perp \neq 0$, and $\|A\|^\perp \doteqdot 0$ if $(E_A)^\perp = 0$. In any case, we have

$$\|A\|^\perp < \|A\| \quad \text{whenever } A \neq I.$$

Corresponding to the decomposition $\mathbb{R}^n = E_A \oplus (E_A)^\perp$, for any vector $x \in \mathbb{R}^n$, we write

$$x = x^E \oplus x^\perp.$$

STEP 3. By Lemma 1.158, for any unit vector $v \in E_A$, $\delta < \pi/4$ with $\varepsilon = \frac{1}{8}\left(\|A\| - \|A\|^\perp\right)$, there exists an element $\beta \in G$, where $\beta(x) = Bx + b$, such that

$$b \neq 0, \quad \left|b^\perp\right| < \left|b^E\right|, \quad \text{and} \quad \|B\| < \frac{1}{8}\left(\|A\| - \|A\|^\perp\right).$$

Since$\|B\| < \frac{1}{8}\|A\| \leq \frac{1}{16} < \frac{1}{2}$, if β is not a pure translation, then $\beta \in S$, so that

$$|b| \geq |a|.$$

Let

$$T \doteqdot \left\{\gamma \in G : c \neq 0, \ \left|c^\perp\right| < \left|c^E\right|, \ \|C\| < \frac{1}{8}\left(\|A\| - \|A\|^\perp\right)\right\}.$$

We may assume that $\beta \in T$ is chosen so that

$$|b| = \min_{\gamma \in T} |\mathrm{trans}(\gamma)| > 0. \tag{1.158}$$

STEP 4. Lemma 1.159 follows from:

$$[\alpha, \beta] \in T \quad \text{and} \quad |\mathrm{trans}([\alpha, \beta])| < |b| \tag{1.159}$$

since this is a contradiction to (1.158) and hence to our assumption that S is nonempty. The rest of the section is devoted to the proof of (1.159).

Since $\|A\| \leq \frac{1}{2}$, we have

$$\|\mathrm{rot}\,([\alpha,\beta])\| = \|[A,B]\| \leq 2\,\|A\| \cdot \|B\| \leq \|B\| \leq \frac{1}{8}\left(\|A\| - \|A\|^{\perp}\right).$$

Now consider

$$\begin{aligned}\mathrm{trans}\,([\alpha,\beta]) &= (A-I)\,b + (I-[A,B])\,b + A\,(I-B)\,A^{-1}a \\ &= (A-I)\,b^{E} + (A-I)\,b^{\perp} + r\end{aligned}$$

where

$$r \doteqdot (I-[A,B])\,b + A\,(I-B)\,A^{-1}a.$$

If β is not a translation, then $\beta \in S$ and $|b| \geq |a|$. This implies that

$$|r| \leq (\|[A,B]\| + \|B\|)\,|b|.$$

Since $\|[A,B]\| \leq \|B\|$, we have

$$|r| \leq 2\,\|B\| \cdot |b|.$$

Since $\left|b^{\perp}\right| < \left|b^{E}\right|$, we have

$$|r| < 4\,\|B\| \cdot \left|b^{E}\right| \leq \frac{1}{2}\left(\|A\| - \|A\|^{\perp}\right)\left|b^{E}\right|. \tag{1.160}$$

The above inequality also holds when β is a translation since then $B = I$, which implies $r = 0$ (and the RHS of (1.160) is positive). We conclude that

$$\begin{aligned}\left|\mathrm{trans}\,([\alpha,\beta])^{\perp}\right| &= \left|(A-I)\,b^{\perp} + r^{\perp}\right| \\ &\leq \|A\|^{\perp}\left|b^{\perp}\right| + \frac{1}{2}\left(\|A\| - \|A\|^{\perp}\right)\left|b^{E}\right| \\ &< \frac{1}{2}\left(\|A\| + \|A\|^{\perp}\right)\left|b^{E}\right|\end{aligned}$$

and

$$\begin{aligned}\left|\mathrm{trans}\,([\alpha,\beta])^{E}\right| &= \left|(A-I)\,b^{E} + r^{E}\right| \\ &\geq \|A\| \cdot \left|b^{E}\right| - \left|r^{E}\right| \\ &> \|A\| \cdot \left|b^{E}\right| - \frac{1}{2}\left(\|A\| - \|A\|^{\perp}\right)\left|b^{E}\right| \\ &= \frac{1}{2}\left(\|A\| + \|A\|^{\perp}\right)\left|b^{E}\right|.\end{aligned}$$

Therefore

$$\left|\mathrm{trans}\,([\alpha,\beta])^{\perp}\right| < \left|\mathrm{trans}\,([\alpha,\beta])^{E}\right|,$$

so that $[\alpha, \beta] \in T$. Finally

$$\begin{aligned}
|\text{trans}([\alpha, \beta])| &= |(A - I)\, b + r| \\
&\leq \|A\| \cdot |b| + |r| \\
&< \frac{1}{2}\,|b| + \frac{1}{2}\left(\|A\| - \|A\|^{\perp}\right)\left|b^E\right| \\
&\leq \frac{3}{4}\,|b| < |b|\,.
\end{aligned}$$

Thus we have (1.159) and the proof is complete. □

15. Compendium of inequalities

In this section we list a few of the inequalities that are used somewhere in the book.

For all $x \in \mathbb{R}$, $e^x \geq 1 + x$.

Arithmetic-geometric mean inequality: If $x_1, x_2, \ldots, x_n \geq 0$, then

$$\frac{x_1 + x_2 + \cdots + x_n}{n} \geq (x_1 x_2 \cdots x_n)^{1/n}$$

with equality if and only $x_1 = x_2 = \cdots = x_n$.

The L^p**-norm** of a measurable function is defined by

$$\|f\|_p \doteqdot \left(\int_M |f|^p \, d\mu\right)^{1/p}.$$

Hölder's inequality: If $p, q \in [1, \infty]$ are such that $\frac{1}{p} + \frac{1}{q} = 1$ and if $f \in L^p(M)$ and $h \in L^q(M)$, then

$$\int_M |fh| \, d\mu \leq \|f\|_p \, \|h\|_q$$

and equality holds if and only if $a\,|f|^p = b\,|h|^q$ a.e. for some constants a and b not both zero.

Jensen's inequality: If $\varphi : \mathbb{R} \to \mathbb{R}$ is convex and $f \in L^1(M^n)$, then

$$\frac{1}{\operatorname{Vol}(g)} \int_{M^n} \varphi \circ f \; d\mu \geq \varphi\left(\frac{1}{\operatorname{Vol}(g)} \int_{M^n} f \; d\mu\right).$$

EXERCISE 1.161. *Prove that if* $\{f_i\}_{i=1}^k$ *and* $\{p_i\}_{i=1}^k$ *are sequences of non-negative numbers with* $p_1 + \cdots + p_k = 1$, *then*

$$f_1^{p_1} \cdots f_1^{p_k} \leq p_1 f_1 + \cdots + p_k f_k$$

where $0^0 \doteqdot 1$. *Derive Jensen's inequality from this.*[8]

EXERCISE 1.162. *Work out the statement of Jensen's inequality when* $\varphi(x) = x \log x$.

[8] See also §25 of [**178**].

L^2 **Sobolev inequality**: If $\int_{M^n} \varphi^2 d\mu = 1$, where $n > 2$, then

$$C_s(g)\left[\int_{M^n} \varphi^{\frac{2n}{n-2}} d\mu\right]^{\frac{n-2}{n}} \leq \int_{M^n} |\nabla\varphi|^2 d\mu + \operatorname{Vol}(g)^{-2/n}$$

where $C_s(g) > 0$ depends only on g.

REMARK 1.163. *If we consider a power function of the distance function to a point (such as the origin): $\varphi(x) = |x|^p$, then the borderline exponent for $\varphi \in W^{1,2}_{\text{loc}}$ is $p = \frac{2-n}{2}$. This is also the borderline exponent for $\varphi \in L^{2n/(n-2)}_{\text{loc}}$.*

Logarithmic Sobolev inequality: For any $a > 0$, there exists a constant $C(a, g)$ such that if $\varphi > 0$ satisfies $\int_{M^n} \varphi^2 d\mu = 1$, then

$$\int_{M^n} \varphi^2 \log\varphi d\mu \leq a \int_{M^n} |\nabla\varphi|^2 d\mu + C(a, g).$$

REMARK 1.164. *Note that if $\varphi(x) = e^{-|x|^2/2}$, then $|\nabla\varphi|^2 = |x|^2 e^{-|x|^2} = -2\varphi^2 \log\varphi$.*

16. Notes and commentary

§1. Additional references on the geometry and topology of 3-manifolds include the books by Benedetti-Petronio [**43**], Hatcher [**278**], Hempel [**280**], Jaco [**304**], Kapovich [**307**], Morgan-Bass [**387**], and Thurston [**488**].

§2. There are several good books on Riemannian geometry including Chavel [**98**], Cheeger-Ebin [**103**], do Carmo [**183**], Eisenhart [**198**], Gallot-Hulin-Lafontaine [**218**], Gromov [**242**], Hicks [**282**], Kobayashi-Nomizu [**327**], O'Neill [**410**], Petersen [**423**], and for bigger game, the five volume Spivak [**478**].

The notion of connection extends to arbitrary vector bundles as follows.

DEFINITION 1.165. *A **linear connection** D on a vector bundle E over a differentiable manifold M^n is an $\mathbb{R}$-linear map:*

$$D : C^\infty(E \otimes TM) \to C^\infty(E),$$

satisfying the product rule:

$$D_X(f \cdot s) = f \cdot D_X s + Xf \cdot s,$$

where we used the notation: $D_X s = D(s \otimes X)$.

If (M^n, g) is a Riemannian manifold, the connection D induces linear connections, also denoted by D, on the bundles of E-valued differential forms $\wedge^p T^*M \otimes E$. Let $\Omega^p(E) = C^\infty(\wedge^p T^*M \otimes E)$. The exterior covariant derivative d^D acting on E-valued differential forms is then defined as follows:

$$d^D : \Omega^p(E) \to \Omega^{p+1}(E),$$

where

$$d^D \alpha(X_0, \ldots, X_p) \doteqdot \sum_{j=0}^{p} (-1)^j D_{X_j} \alpha(X_0, \ldots, X_{j-1}, X_{j+1}, \ldots, X_p).$$

In local coordinates, we write this as:

$$\left(d^D\alpha\right)_{i_0\cdots i_p} = \sum_{j=0}^{p}(-1)^j D_{i_j}\alpha_{i_0\cdots i_{j-1}i_{j+1}\cdots i_p}.$$

The Riemann curvature tensor is a section of the bundle $\wedge^2 M\otimes T^*M\otimes TM$, or equivalently, $\mathrm{Rm}\in\Omega^2(T^*M\otimes TM)$. Hence the second Bianchi identity may be rephrased as:

$$0 = d^\nabla \mathrm{Rm}\in\Omega^3(T^*M\otimes TM),$$

where d^∇ denotes exterior covariant derivative using the connection on $T^*M\otimes TM$ induced by the Levi-Civita connection.

§3. There are many references for the Laplacian and Hessian comparison theorems. We note in particular the books by Schoen-Yau [**447**], P. Li [**340**] and the lectures of Schoen [**443**]. Most of the presentation in this section follows [**447**].

The fact that a locally Lipschitz function is a.e. C^1 is a special case of the following more general result (see Theorem 3 on p. 250 of [**479**]). Let (M^n,g) be a Riemannian manifold and let f be a function on a open set $U\subset M^n$. Then f is C^1 a.e. in U if and only if for a.e. $x\in U$

$$f(y)-f(x) = O(d(x,y)) \qquad \text{as } d(x,y)\to 0.$$

§4. Exercise 1.92 is equivalent to:

EXERCISE 1.166. *Use the Cartan structure equations to compute the connection 1-forms and curvatures for the metric*

$$g = h(r)^2\,dr^2 + f(r)^2\,g_{S^{n-1}}.$$

REMARK 1.167. *By making the substitution $ds = h(r)\,dr$ we see that this is equivalent to Exercise 1.92.*

SOLUTION. Let $\{\bar e_i\}_{i=1}^{n-1}$ and $\left\{\bar\omega^i\right\}_{i=1}^{n-1}$ be local orthonormal frame and coframe fields for $\left(S^{n-1},g_{S^{n-1}}\right)$ and let $\bar\omega_i^j$ be the corresponding connection 1-forms which satisfy the first and second structure equations:

$$d\bar\omega^i = \bar\omega^j\wedge\bar\omega_j^i$$
$$\overline{\mathrm{Rm}}_j^i = d\bar\omega_j^i - \bar\omega_i^k\wedge\bar\omega_k^j.$$

For g define the orthonormal frame $e_n \doteqdot \frac{1}{h(r)}\frac{\partial}{\partial r}$, $e_i\doteqdot\frac{1}{f(r)}\bar e_i$ and the orthonormal coframe $\omega^n\doteqdot h(r)\,dr$, $\omega^i\doteqdot f(r)\,\bar\omega^i$, for $i=1,\ldots,n-1$. By Exercise 1.39 we have

$$\omega_i^k(e_j) = \frac{1}{2}\left(d\omega^i(e_j,e_k)+d\omega^j(e_i,e_k)-d\omega^k(e_j,e_i)\right).$$

Using this and $d\omega^n=0$ we compute for $i,k\le n-1$

$$\omega_i^k = \bar\omega_i^k \tag{1.161}$$

$$\omega_n^k = \frac{e_n f}{f} \omega^k. \tag{1.162}$$

Now by the second structure equation we have

$$\begin{aligned}\operatorname{Rm}(g)_i^{\ j} &= d\omega_i^j - \sum_{k=1}^{n-1} \omega_i^k \wedge \omega_k^j - \omega_i^n \wedge \omega_n^j \\ &= \frac{1 - (e_n f)^2}{f^2} \omega^j \wedge \omega^i\end{aligned}$$

and

$$\begin{aligned}\operatorname{Rm}(g)_n^{\ j} &= d\omega_n^j - \sum_{k=1}^{n-1} \omega_n^k \wedge \omega_k^j \\ &= d\left(\frac{e_n f}{f}\right) \wedge \omega^j + \frac{e_n f}{f}\left(d\omega^j - \omega^k \wedge \omega_k^j\right) \\ &= -\frac{e_n e_n f}{f} \omega^j \wedge \omega^n.\end{aligned}$$

§5. [**103**] is an excellent reference for the first and second variation formulas and more generally, comparison geometry.

§5.3. We say that a C^0 function $\phi : M \to \mathbb{R}$ is **convex** if for every constant speed geodesic $\gamma : [0,1] \to M$, we have

$$\phi(\gamma(s)) \leq (1-s)\,\phi(\gamma(0)) + s\phi(\gamma(1))$$

for all $s \in [0,1]$.

EXERCISE 1.168. *Show that if ϕ is a C^2 function, then $\nabla_i \nabla_j \phi \geq 0$ if and only if ϕ is convex.*

LEMMA 1.169. *Let (M^n, g) be a complete Riemannian manifold and let $\phi : M \to \mathbb{R}$ be a C^2 convex function. Then a point is critical point of ϕ if and only if it is a global minimum of ϕ.*

PROOF. Suppose that $p \in M$ is a critical point of ϕ. Given any point $x \in M$, let $\gamma : [0,1] \to M$ be a geodesic joining p to x (since (M^n, g) is complete a minimal such geodesic exists). We have

$$\begin{aligned}\frac{d^2}{du^2}\phi(\gamma(u)) &= \frac{d}{du}\langle \nabla\phi, \dot\gamma\rangle \\ &= \langle \nabla\nabla\phi, \dot\gamma \otimes \dot\gamma\rangle \geq 0\end{aligned}$$

(we used $\nabla_{\dot\gamma}\dot\gamma = 0$.) Since $\frac{d}{du}\big|_{u=0}\langle \nabla\phi, \dot\gamma\rangle = 0$, we have

$$\frac{d}{du}\langle \nabla\phi, \dot\gamma\rangle \geq 0$$

for all $u \in [0,1]$. Hence $\phi(x) \geq \phi(p)$. □

In our study of Ricci flow, we shall find it useful to make comparisons with the mean curvature flow of hypersurfaces and other geometric flows. For this reason we recall here some basic facts about hypersurfaces. Let $X : M^n \to \mathbb{R}^{n+1}$ be a smooth immersed hypersurface and N a choice of unit normal vector to $X(M^n)$ (actually N is a vector field with values in $\mathbb{R}^{n+1}$ along M^n.) When $X(M^n)$ is embedded closed we shall usually take N to be the outward unit normal. The **first fundamental form** is defined by

$$g(V, W) \doteqdot \langle V, W \rangle$$

for $V, W \in T_x X(M^n)$, where $\langle \ , \ \rangle$ denotes the euclidean inner product. More correctly (the distinction should be made in the immersed and nonembedded case)

$$g(V, W) \doteqdot \langle X_* V, X_* W \rangle$$

for $V, W \in T_p M^n$. Recall that the second fundamental form is defined by (1.53):

$$h(V, W) \doteqdot \langle D_V N, W \rangle = -\langle D_V W, N \rangle$$

or $V, W \in T_x X(M^n)$, where D is the euclidean covariant derivative, i.e., the directional derivative. To get the second equality in the line above, we extend W to a tangent vector field in a neighborhood of x and used $\langle N, W \rangle \equiv 0$. The second fundamental form measures the extrinsic curvature of the the hypersurface $X(M^n)$, i.e., how the normal is changing. We have the Gauss equations (1.54)

$$\langle \mathrm{Rm}(U, V) W, Y \rangle = h(U, Y) h(V, W) - h(U, W) h(V, Y)$$

and the Codazzi equations (see Exercise 1.41)

$$(\nabla_U h)(V, W) = (\nabla_V h)(U, W). \tag{1.163}$$

The mean curvature is the trace of the second fundamental form:

$$H \doteqdot \sum_{i=1}^{n} h(e_i, e_i) = g^{ij} h_{ij}$$

where in the first expression $\{e_i\}$ is an orthonormal frame on $X(M^n)$ and in the second expression in terms of a local coordinate system $\{x^i\}$ in M^n, which gives local coordinates $\{x^i \circ X^{-1}\}$ on $X(M^n)$,

$$g_{ij} \doteqdot g\left(X_* \frac{\partial}{\partial x^i}, X_* \frac{\partial}{\partial x^j}\right) \qquad h_{ij} \doteqdot h\left(X_* \frac{\partial}{\partial x^i}, X_* \frac{\partial}{\partial x^j}\right)$$

and $g^{ij} \doteqdot (g^{-1})_{ij}$. The Ricci curvature of g may be expressed as

$$\mathrm{Rc}(U, V) = H h(U, V) - \sum_{i=1}^{n} h(U, e_i) h(V, e_i)$$

or in shorthand notation, $\mathrm{Rc} = Hh - h^2$. The scalar curvature is $R = H^2 - |h|^2$. The **Gauss curvature** is defined by

$$K \doteqdot \frac{\det h}{\det g}.$$

The **principal curvatures** $\kappa_1, \ldots, \kappa_n$ are the eigenvalues of h with respect to g. We have

$$H = \kappa_1 + \cdots + \kappa_n, \qquad R = \sum_{i \neq j} \kappa_i \kappa_j, \qquad K = \kappa_1 \cdot \cdots \cdot \kappa_n.$$

Analogous formulas hold for the m**th mean curvatures**, which may be defined as mth elementary symmetric function of the principal curvatures.

EXERCISE 1.170. *Using the Codazzi equations, show that if a hypersurface $M^n \subset \mathbb{R}^{n+1}$, $n \geq 2$, is totally umbillic, that is, $h = \varphi g$ for some function φ (tracing we see that $\varphi = \frac{1}{n} H$), then the mean curvature is constant.*

EXERCISE 1.171 (Weyl's Estimate). *Let $M^2 \subset \mathbb{R}^3$ be a convex surface, that is, $h > 0$. Show that*

$$\Box H = \Delta \log K + |\nabla_i h_{jk}|^2_{g,h,h} + H^2 - 4K$$

where $\Box \doteqdot \left(h^{-1}\right)^{ij} \nabla_i \nabla_j$ and $|a_{ijk}|^2_{g,h,h} \doteqdot g^{ip} \left(h^{-1}\right)^{jq} \left(h^{-1}\right)^{kr} a_{ijk} a_{pqr}$. Note that by the arithmetic-geometric mean inequality, $H^2 - 4K \geq 0$. Applying the maximum principle at a point x where H attains its maximum, we have $\Box H(x) \leq 0$ so that

$$H(x)^2 \leq 4K(x) - \Delta \log K(x).$$

Note that the RHS *depends only on the intrinsic geometry of the surface:*

$$\max_M H^2 \leq C(g)$$

where $C(g)$ depends only on the first fundamental form g. Since $|h|^2 = H^2 - 2K \leq H^2$, we also obtain a bound for $|h|$.

§11. A couple of more recent references on the determinant of elliptic operators such as the Laplacian are [**89**], [**90**].

CHAPTER 2

Elementary aspects of the Ricci flow equation

1. Some geometric flows predating Ricci flow

Geometric flows have been around at least since Mullins paper [**393**] in 1956 proposing the curve shortening flow to model the motion of idealized grain boundaries. In 1964 the seminal paper of Eells and Sampson [**196**] introduced the harmonic map heat flow and used it prove the existence of harmonic maps into targets with nonpositive sectional curvature. In 1974 Firey [**210**] proposed the Gauss curvature flow to model the shapes of worn stones and considered the case where the surface is invariant under minus the identity. In 1975 Hamilton [**254**] continued the study of the harmonic map heat flow by considering manifolds with boundary. In 1978 Brakke [**55**] studied the mean curvature flow and proved regularity properties for it.

We begin by giving a baby introduction to how Ricci flow may be used to approach the geometrization conjecture. The intent of this section is only to give the reader some preliminary intuition about what type of results one would like to prove about Ricci flow in dimension 3. Next we describe some of the evolution equations for the curvatures of solutions to Ricci flow. Since the scalar curvature satisfies a particularly simple equation, we start with its evolution. This leads to the heat equation, for which a principal tool is the maximum principle, which we discuss both when the manifold is compact and noncompact. Although the Ricci flow is not exactly variational in the usual sense, we consider the Einstein-Hilbert functional and see that Ricci flow is similar to a gradient flow. In Volume 2 we shall see a precise way in which Ricci flow is a gradient flow, due to Perelman. We then discuss some local coordinate calculations which lead to variation formulas for the curvatures. Applying this to the Ricci tensor and its modification, we obtain DeTurck's trick, which is used to prove the short time existence of solutions of Ricci flow on closed manifolds.

2. Ricci flow and geometrization: a short preview

In this section we give a very rough and intuitive description of Hamilton's idea for using Ricci flow to approach Thurston's Geometrization Conjecture. In the following discussion all 3-manifolds will be closed and orientable. Start with a closed Riemannian manifold $\left(M_1^3, g_0\right)$. We want to infer the existence of a geometric decomposition for M_1^3 by studying the properties of a solution $\left(M^3(t), g(t)\right)$ of the **Ricci flow with surgeries** with $M^3(0) = M_1^3$ and $g(0) = g_0$. By Ricci flow with surgeries we mean a sequence of solutions $\left(M_i^3, g_i(t)\right)$, $t \in [\tau_{i-1}, \tau_i]$, with $\tau_{i-1} < \tau_i \leq \infty$ and $\tau_0 = 0$, of Ricci flow where $\left(M_{i+1}^3, g_{i+1}(\tau_i)\right)$ is obtained from $\left(M_i^3, g_i(\tau_i)\right)$ by a geometric-topological surgery. $M^3(t) = M_i^3$ and $g(t) = g_i(t)$ for $t \in [\tau_{i-1}, \tau_i]$ so that the manifold and metric are doubly defined at the surgery times τ_i.

The short and long time existence theorems imply that there exists a unique maximal solution to the Ricci flow $\left(M_1^3, g_1(t)\right)$, $t \in [0, T_1)$, where either

$$\sup_{M_1^3 \times [0,T_1)} |\mathrm{Rm}(g_1)| = \infty$$

or $T_1 = \infty$. If $T_1 < \infty$, then we want to prove that in high curvature regions (near the singularity time) the metric looks like part of an $S^2 \times \mathbb{R}$ cylinder (this is not quite true, but for the sake of simplicity let's assume this for now). At some appropriate time $\tau_1 \lessapprox T_1$ right before the singularity forms we perform a geometric-topological surgery by cutting out an $S^2 \times B^1$ from the cylindrical region, which has two disjoint 2-spheres as its boundary: $\partial\left(S^2 \times B^1\right) \cong S^2 \times S^0 \cong \partial\left(B^3 \times S^0\right)$, and replacing it by two balls: $B^3 \times S^0$.[1] We call the new manifold $\left(M_2^3, g_1\right)$. Let $\left(M_2^3, g_2(t)\right)$, $t \in [\tau_1, T_2)$, be the unique maximal solution to the Ricci flow with $g_2(\tau_1) = g_1$. Topologically, M_2^3 and M_1^3 are related as follows. If $M_2^3 \cong M_{2,1}^3 \sqcup M_{2,2}^3$ is the disjoint union of two connected manifolds, then $M_1^3 \cong M_{2,1}^3 \# M_{2,2}^3$. On the other hand, if M_2^3 is connected, then $M_1^3 \cong M_2^3 \# \left(S^2 \times S^1\right)$. The reason for this is reversing the surgery process we see that M_1^3 is obtained from M_2^3 by attaching a handlebody. Isotopying this handlebody so that it is attached to two 2-spheres inside a 3-ball in M_2^3 we see that M_1^3 is the connected sum of M_2^3 with an orientable S^2 bundle over S^1 which must be $S^2 \times S^1$.

Now repeat the process so that if $T_2 < \infty$, then we perform a geometric-topological surgery at some time $\tau_2 \lessapprox T_2$ to get $\left(M_3^3, g_2\right)$, etc. As long as $T_i < \infty$ for all i, we have surgery times $\tau_i \lessapprox T_i$. Hence, we either obtain a finite sequence of solutions $\left(M_i^3, g_i(t)\right)$, $t \in [\tau_{i-1}, \tau_i]$, where $1 \leq i \leq I < \infty$ where the last solution $\left(M_I^3, g_I(t)\right)$ is defined for $t \in [\tau_{I-1}, \infty)$, or we obtain an infinite sequence of solutions. In the latter case it is conceivable that the sequence of surgery times $\{\tau_i\}$ accumulates to a finite time. This is a major case which needs to be ruled out. Barring this, we obtain a solution of the Ricci flow with surgeries, consisting of a countable (infinite or finite) sequence of solutions of the Ricci flow, which is defined for all time $t \in [0, \infty)$. Next we want to infer the existence of a geometric structure on M_i^3 for i large enough via a sufficient understanding of the geometric properties of a metric $g_i(t_i)$ where $t_i \in [\tau_{i-1}, \tau_i]$. In particular, we want to show that the manifold may be decomposed into pieces with incompressible tori boundary

[1] In Perelman's approach, the surgery is performed *at* the singularity time on the singular manifold limit.

whose interiors either admit complete, finite volume hyperbolic metrics or are collapsed with bounded curvature in the sense of Cheeger-Gromov.

3. Ricci flow and the evolution of scalar curvature

Given a 1-parameter family of metrics $g(t)$ on a Riemannian manifold M^n, defined on a time interval $\mathcal{I} \subset \mathbb{R}$, Hamilton's **Ricci flow equation** is

$$\frac{\partial}{\partial t} g_{ij} = -2R_{ij}.$$

For any C^∞ metric g_0 on a closed manifold M^n, there exists a unique solution $g(t)$, $t \in [0, \epsilon)$, to the Ricci flow equation for some $\varepsilon > 0$, with $g(0) = g_0$. This was proved in [**255**] and shortly thereafter a much simpler proof was given by DeTurck [**180**] (see [**181**] for an improved version). We shall briefly discuss short time existence in section 8. For any C^∞ complete metric g_0 with bounded sectional curvature on a noncompact manifold M^n, such a short time existence result for solutions to the Ricci flow was proved by W.-X. Shi [**462**].

Definition 2.1 (Complete solution). *A solution $g(t)$, $t \in \mathcal{I}$, of the Ricci flow is said to be* ***complete*** *if for each $t \in \mathcal{I}$, the Riemannian metric $g(t)$ is complete.*

Definition 2.2 (Bounded curvature solution). *We say that a solution $g(t)$, $t \in \mathcal{I}$, of the Ricci flow has* ***bounded curvature*** *if on every compact subinterval $[a, b] \subset \mathcal{I}$ the Riemann curvature tensor is bounded. In particular, we do not assume the curvature bound is uniform in time on noncompact time intervals.*

Problem 2.3 (Uniqueness of Ricci flow on noncompact manifolds). *Under what conditions does uniqueness hold for complete solutions to the Ricci flow on noncompact manifolds? A reasonable assumption may be bounded curvature.*

Given that we have short time existence, we are interested in the long time behavior of the solution. Toward this end we want to derive the evolution equations for the Riemann, Ricci and scalar curvatures

$$\frac{\partial}{\partial t} R_{ijk\ell}, \ \frac{\partial}{\partial t} R_{ij}, \ \frac{\partial}{\partial t} R$$

as well as other geometric quantities.

First we look at some simple examples. Let $M^n = S^n$ and g_{S^n} denote the standard metric on the unit n-sphere in euclidean space. If $g_0 = r_0^2 g_{S^n}$ for some $r_0 > 0$ (r_0 is the radius), then

$$g(t) \doteqdot \left(r_0^2 - 2(n-1)t\right) g_{S^n} \tag{2.1}$$

is a solution to the Ricci flow with $g(0) = g_0$ defined on the maximal time interval $(-\infty, T)$, where $T \doteqdot r_0^2 / 2(n-1)$. That is, under the Ricci flow, the sphere stays round and shrinks at a steady rate.

EXERCISE 2.4 (Standard shrinking sphere). *Show that the metrics defined by (2.1) is a solution to the Ricci flow.* HINT*: Use the scale-invariance of Ricci (Exercise 1.11) and* $\operatorname{Rc}(g_{S^n}) = (n-1)\, g_{S^n}$*. Show that the scalar curvature of the solution is given by*

$$R(g(t)) = \frac{n(n-1)}{r_0^2 - 2(n-1)t}.$$

In particular, the solution with $r_0^2 = n(n-1)$*, which is defined on* $(-\infty, n/2)$*, has scalar curvature* $R(g(t)) = \frac{1}{1-2t/n}$*.*

EXERCISE 2.5 (Homothetic Einstein solutions). *Suppose that* g_0 *is an Einstein metric, i.e.,* $\operatorname{Rc}(g_0) \equiv c g_0$ *for some* $c \in \mathbb{R}$*. Derive the explicit formula for the solution* $g(t)$ *of the Ricci flow with* $g(0) = g_0$*. Observe that* $g(t)$ *is homothetic to the initial metric* g_0 *and shrinks, is stationary, or expands depending on whether* c *is positive, zero, or negative, respectively.*

EXERCISE 2.6 (Product solutions). *Suppose that* $(M_1, g_1(t))$ *and* $(M_2, g_2(t))$ *are solutions of the Ricci flow on a common time interval* $\mathcal{I}$*. Show that* $(M_1 \times M_2, g_1(t) \times g_2(t))$ *is a solution of the Ricci flow. Hint: see Exercise 1.37. In particular, if* $(M^n, g(t))$ *is a solution of the Ricci flow, then so is* $\left(M^n \times \mathbb{R}, g(t) + dr^2\right)$ *(we can replace* $\left(\mathbb{R}, dr^2\right)$ *by any static flat manifold.)*

PROBLEM 2.7. *What explicit solutions of the Ricci flow are there? Of course we have the constant sectional curvature solutions, products and quotients thereof. Some other solutions are the cigar and Rosenau solutions on* $\mathbb{R}^2$ *and* S^2*, respectively (see Chapter 4). It would be interesting to find explicit solutions in dimensions* ≥ 3*.*

We now begin with the **evolution equation for the scalar curvature** since this is the easiest:

$$\boxed{\frac{\partial}{\partial t} R = \Delta R + 2\left|\operatorname{Rc}\right|^2.} \tag{2.2}$$

When $n = 2$, since then $\operatorname{Rc} = \frac{1}{2} R g$, we have

$$\frac{\partial}{\partial t} R = \Delta R + R^2. \tag{2.3}$$

Note the similarity to the heat equation $\frac{\partial u}{\partial t} = \Delta u$. To derive formula (2.2) we recall the **variation of scalar curvature**.

LEMMA 2.8 (Variation of scalar curvature). *If* $\frac{\partial}{\partial s} g_{ij} = v_{ij}$*, then*

$$\boxed{\frac{\partial}{\partial s} R = -\Delta V + \operatorname{div}(\operatorname{div} v) - \langle v, \operatorname{Rc}\rangle,} \tag{2.4}$$

where $V = g^{ij} v_{ij} = \operatorname{trace}(v)$ *is the trace of* v*.*

For the derivation of this formula see (2.25) below. Note that $\operatorname{div}(\operatorname{div} v) = \nabla_p \nabla_q v_{pq}$. Plugging in $v = -2\operatorname{Rc}$ and using the contracted second Bianchi identity $2\nabla_q R_{pq} = \nabla_p R$, we obtain (2.2).

4. The maximum principle for heat-type equations

When we have heat-type equations, we can apply a powerful tool, the **maximum principle**. The maximum principle is basically the first and second derivative tests in calculus. For elliptic equations on a manifold, the facts we use are that if a function $f : M \to \mathbb{R}$ attains its minimum at a point $x_0 \in M$, then

$$\nabla f(x_0) = 0 \text{ and } \Delta f(x_0) \geq 0.$$

For equations of parabolic type, a simple version says the following.

Proposition 2.9 (Weak maximum principle for supersolutions of the heat equation). *Let $g(t)$ be a family of metrics on a closed manifold M^n and let $u : M^n \times [0,T) \to \mathbb{R}$ satisfy*

$$\frac{\partial}{\partial t} u \geq \Delta_{g(t)} u.$$

Then if $u \geq c$ at $t = 0$ for some $c \in \mathbb{R}$, then $u \geq c$ for all $t \geq 0$.

Proof. The idea is simply that given a time $t_0 \geq 0$, if the spatial minimum of u is attained at a point $x_0 \in M$, then

$$\frac{\partial u}{\partial t}(x_0, t_0) \geq \left(\Delta_{g(t)} u\right)(x_0, t_0) \geq 0,$$

so that the minimum should be nondecreasing. Note that at (x_0, t_0) we actually have $(\nabla_i \nabla_j u) \geq 0$ (this is simply the second derivative test from calculus). Here and throughout this book, when we write $(A_{ij}) \geq 0$ for some matrix A we mean that it is nonnegative definite.

More rigorously, we proceed as follows. Given any $\varepsilon > 0$, define $u_\varepsilon : M \times [0,T) \to \mathbb{R}$

$$u_\varepsilon = u + \varepsilon (1 + t).$$

Since $u \geq c$ at $t = 0$, we have $u_\varepsilon > c$ at $t = 0$. Now suppose for some $\varepsilon > 0$ we have $u_\varepsilon \leq c$ somewhere in $M \times (0,T)$. Then since M is closed, there exists $(x_1, t_1) \in M \times (0,T)$ such that $u_\varepsilon(x_1, t_1) = c$ and $u_\varepsilon(x,t) > c$ for all $x \in M$ and $t \in [0, t_1)$. We then have at (x_1, t_1)

$$0 \geq \frac{\partial u_\varepsilon}{\partial t} \geq \Delta_{g(t)} u_\varepsilon + \varepsilon > 0,$$

which is a contradiction. Hence $u_\varepsilon > c$ on $M \times [0,T)$ for all $\varepsilon > 0$ and by taking the limit as $\varepsilon \to 0$ we get $u \geq c$ on $M \times [0,T)$. □

Applying the maximum principle to the equation for the scalar curvature (2.2) yields:

Corollary 2.10 (Lower bound of scalar curvature is preserved under RF). *If $g(t)$, $t \in [0,T)$, is a solution to the Ricci flow on a closed manifold with $R \geq c$ at $t = 0$ for some $c \in \mathbb{R}$, then $R \geq c$ for all $t \in [0,T)$. In particular, nonnegative (positive) scalar curvature is preserved under the Ricci flow.*

A simple extension of Proposition 2.9 that we shall find convenient to use is the following (for the proof, see [**153**], Theorem 4.4 on p. 96).

LEMMA 2.11 (Weak minimum principle - comparing with the ODE). *Suppose $g(t)$ is a family of metrics on a closed manifold M^n and $u : M^n \times [0,T) \to \mathbb{R}$ satisfies*

$$\frac{\partial}{\partial t} u \leq \Delta_{g(t)} u + \langle X(t), \nabla u \rangle + F(u),$$

where $X(t)$ is a time-dependent vector field and F is a Lipschitz continuous function. If $u \leq c$ at $t = 0$ for some $c \in \mathbb{R}$, then $u(x,t) \leq U(t)$ for all $x \in M^n$ and $t \geq 0$, where $U(t)$ is the solution to the ODE $\frac{dU}{dt} = F(U)$ *with $U(0) = c$.*

EXERCISE 2.12. *Prove Lemma 2.11. Hint: Use*

$$\frac{\partial}{\partial t}(u - U) \leq \Delta_{g(t)}(u - U) + \langle X(t), \nabla(u - U) \rangle + F(u) - F(U)$$

and the Lipschitz property of F.

REMARK 2.13. *In the statement of Lemma 2.11 we may reverse the signs on all of the inequalities (except $t \geq 0$) to obtain that a supersolution to a semi-linear heat equation is bounded below by any solution to the corresponding* ODE *with initial value less than or equal to the infimum of the initial value of the supersolution of the* PDE.

Since $|\mathrm{Rc}|^2 \geq \frac{1}{n} R^2$ (more generally, $|a|_g^2 \geq \frac{1}{n} \mathrm{Trace}_g(a)^2$ for any 2-tensor a ; see Exercise 2.15 below), equation (2.2) implies

$$\frac{\partial}{\partial t} R \geq \Delta R + \frac{2}{n} R^2. \tag{2.5}$$

Since the solutions to the ODE $\frac{d\rho}{dt} = \frac{2}{n}\rho^2$ are $\rho(t) = \frac{n}{n\rho(0)^{-1} - 2t}$, the lemma implies that whenever the maximum principle applies, one has

$$\boxed{R(x,t) \geq \frac{n}{n\left(\inf_{t=0} R\right)^{-1} - 2t}} \tag{2.6}$$

for all $x \in M^n$ and $t \geq 0$. It $\rho(0) > 0$, then $\rho(t)$ tends to infinity in finite time. Hence

COROLLARY 2.14 (Finite singularity time for positive scalar curvature). *If (M^n, g_0) is a closed Riemannian manifold with positive scalar curvature, then for any solution $g(t)$, $t \in [0,T)$ to the Ricci flow with $g(0) = g_0$ we have*

$$T \leq \frac{n}{2R_{\min}(0)} < \infty.$$

EXERCISE 2.15 (Norm of 2-tensor dominates trace). *By choosing coordinates where $g_{ij} = \delta_{ij}$ at a point, show that for any 2-tensor a_{ij}*

$$|a_{ij}|_g^2 \geq \frac{1}{n}\left(g^{ij} a_{ij}\right)^2.$$

EXERCISE 2.16. *Let u be a solution to the heat equation with respect to a metric $g(t)$ evolving by the Ricci flow.*

(1) (a) *Show that*

$$\frac{\partial}{\partial t}|\nabla u|^2 = \Delta |\nabla u|^2 - 2|\nabla\nabla u|^2 .$$

(b) *From this deduce*

$$\left(\frac{\partial}{\partial t} - \Delta\right)\left(t|\nabla u|^2 + \frac{1}{2}u^2\right) \leq 0.$$

(c) *Apply the maximum principle to conclude that if M^n is closed, then*

$$|\nabla u| \leq \frac{U}{\sqrt{2}t^{1/2}}$$

where $U \doteqdot \max_{t=0}|u|$ *.*

EXERCISE 2.17. *Let u be a solution to the heat equation on a Riemannian manifold (M^n, g). Show that*

$$\frac{\partial}{\partial t}|\nabla u|^2 = \Delta |\nabla u|^2 - 2|\nabla\nabla u|^2 - 2R_{ij}\nabla_i u \nabla_j u.$$

What estimate for $|\nabla u|$ do you get assuming $\mathrm{Rc} \geq 0$*? When* $\mathrm{Rc} \geq -H$ *for some constant H? Which curvature condition do you need to get decay of $|\nabla u|$ as $t \to \infty$?*

5. The maximum principle on noncompact manifolds

It is well-known that on a complete noncompact manifold (even Euclidean space) the solution to the heat equation $\left(\frac{\partial}{\partial t} - \Delta\right)u(x,t) = 0$ is not unique in general (see (2.46).) This is due to the failure of the maximum principle on noncompact manifolds. However, the maximum principle still holds for solutions satisfying certain growth conditions. We start with the following general result of Karp-Li [**309**]. In Ni-Tam [**404**], a version for metrics evolving by Ricci flow is presented.

Let (M^n, g) be a complete Riemannian manifold.

DEFINITION 2.18. *We say that $u \in H^1_{\mathrm{loc}}(M^n \times [0,T])$ is a* ***weak subsolution of the heat equation*** *if for every nonnegative C^∞ function ϕ with compact support in $M^n \times (0,T)$ we have*

$$\int_0^T \int_{M^n} \left(u\frac{\partial \phi}{\partial t} - \nabla u \cdot \nabla \phi\right) d\mu(x)\, dt \geq 0.$$

Note that if u is C^2, then since ϕ has compact support, we can integrate by parts to get

$$\int_0^T \int_{M^n} \phi\left(\Delta u - \frac{\partial u}{\partial t}\right) d\mu(x)\, dt \geq 0,$$

which implies $\frac{\partial u}{\partial t} \leq \Delta u$. Let $u_+ \doteqdot \max\{0, u\}$ and $d(x, O)$ denote the distance function of x to a fixed point $O \in M^n$.

THEOREM 2.19. *If u is a weak subsolution of the heat equation on $M^n \times [0, T]$ with $u(\cdot, 0) \leq 0$ and if*

$$\int_0^T \int_{M^n} \exp(-\alpha d^2(x, O)) u_+^2(x, t)\, d\mu\,(x)\; dt < \infty \tag{2.7}$$

for some $\alpha > 0$, then $u \leq 0$ on $M^n \times [0, T]$.

Applying the volume comparison theorem we have the following.

COROLLARY 2.20. *If $\mathrm{Rc}(x) \geq -d^2(x, O)$, and $u(x, t)$ is a bounded subsolution, then $u(x, t) \leq 0$ provided $u(x, 0) \leq 0$. In particular, bounded solutions are unique.*

PROOF. A direct application of the volume comparison theorem shows that

$$\mathrm{Vol}\,(B\,(O, r)) \leq \exp\left(ar^2\right)$$

for some $a = a\,(n) > 0$. It is then easy to see that the assumption of Theorem 2.19 holds for some suitably chosen α. $\square$

PROOF OF THEOREM. Let $h(x, t) = -\frac{d^2(x,O)}{4(2\tau - t)}$, which is a Lipschitz function defined on $M^n \times [0, 2\tau)$, and where $\tau > 0$ is to be later chosen sufficiently small. Since $|\nabla d\,(\cdot, O)| = 1$ it follows that

$$|\nabla h|^2 + \frac{\partial h}{\partial t} = 0 \tag{2.8}$$

in the weak sense. Let $0 \leq \varphi_s(x) \leq 1$ be a cut-off function which is 1 inside $B(O, s)$ and compactly supported in $B(O, s+1)$ with $|\nabla \varphi_s| \leq 2$. We have that u_+ also satisfies

$$\left(\frac{\partial}{\partial t} - \Delta\right) u_+ \leq 0$$

in the weak sense. Multiplying the above differential inequality by $\varphi_s^2 e^h u_+$ and integrating by parts, we have that

$$\begin{aligned}
0 &\geq \int_0^\tau \int_{M^n} \varphi_s^2 e^h u_+ \left(\frac{\partial}{\partial t} - \Delta\right) u_+ \, d\mu\, dt \\
&= \int_0^\tau \int_{M^n} e^h \left(\varphi_s^2 |\nabla u_+|^2 + 2\left\langle \nabla \varphi_s, \nabla u_+\right\rangle \varphi_s u_+ + \varphi_s^2 u_+ \left\langle \nabla h, \nabla u_+ \right\rangle\right) d\mu\, dt \\
&\quad + \frac{1}{2}\int_0^\tau \int_{M^n} \varphi_s^2 e^h \left(\frac{\partial}{\partial t} u_+^2\right) d\mu\, dt \\
&\geq \int_0^\tau \int_{M^n} e^h \left(-2|\nabla \varphi_s|^2 u_+^2 - \frac{1}{2}\varphi_s^2 u_+^2 |\nabla h|^2\right) d\mu\, dt + \frac{1}{2}\int_{\mathcal{M}} \varphi_s^2 e^h u_+^2 \Big|_0^\tau \\
&\quad - \frac{1}{2}\int_0^\tau \int_{M^n} \varphi_s^2 e^h u_+^2 \frac{\partial h}{\partial t}\, d\mu\, dt.
\end{aligned}$$

Here we used the Cauchy-Schwarz inequality. Applying the equality (2.8) satisfied by h, we have

$$\int_{M^n} \varphi_s^2 e^h u_+^2 \Big|_\tau \leq 4 \int_0^\tau \int_{M^n} e^h u_+^2 |\nabla \varphi_s|^2 \, d\mu \, dt.$$

Since $h(x,t) = -\frac{d^2(x,O)}{4(2\tau - t)} \leq -\frac{d^2(x,O)}{8\tau}$, if we chose $\tau \leq \frac{1}{8\alpha}$, then we have

$$\int_{M^n} \varphi_s^2 e^h u_+^2 \Big|_\tau \leq 8 \int_0^\tau \int_{B(O,s+1)\setminus B(O,s)} \exp\left(-\alpha d^2(x,O)\right) u_+^2 \, d\mu \, dt.$$

By our assumption (2.7), the right hand side tends to zero as $s \to \infty$, and we conclude that $u_+ \equiv 0$ a.e. on $N^n \times [0,\tau]$. The result follows from iterating the above argument. □

REMARK 2.21. *In* [**346**], *Li-Yau proved the uniqueness of solutions bounded from below under a certain lower bound assumption on the Ricci curvature. The key idea is that one can obtain growth control of positive solutions to the heat equation by their gradient estimates (also called Li-Yau inequalities).*

LEMMA 2.22. *Assume that the curvatures and their first derivatives of* $(M^n, g(t))$, $t \in [0,T)$, *are uniformly bounded. For any* $a > 0$ *and* $A > 0$, *there exists a positive function* $\phi(x,t)$ *and* $b > 0$ *such that*

$$\left(\frac{\partial}{\partial t} - \Delta\right) \phi \geq A\phi \tag{2.9}$$

on $M^n \times [0,T)$ *and*

$$\exp(a \cdot d(O,x)) \leq \phi(x,t) \leq \exp(b \cdot d(O,x)).$$

PROOF. Recall from Theorem 3.6 of [**464**] that there exists a smooth function $f(x)$ and a constant $C_1 > 0$ satisfying

$$C_1^{-1}(1 + d_{g(0)}(O,x)) \leq f(x) \leq C_1(1 + d_{g(0)}(O,x))$$

and

$$|\nabla f|_{g(0)} + |\nabla_{g(0)} \nabla f|_{g(0)} \leq C_1.$$

Since the Ricci tensor is assumed to be uniformly bounded, we have $g(t) \geq cg(0)$ and

$$cd_{g(0)}(x,O) \leq d_{g(t)}(x,O) \leq c^{-1} d_{g(0)}(x,O)$$

for some $c > 0$. By the first derivative of curvature bound, we also have $\left|\Gamma_{ij}^k(t) - \Gamma_{ij}^k(0)\right| \leq C_2$. Hence there exists $C < \infty$ such that

$$|\nabla f|_{g(t)} + |\nabla_{g(t)} \nabla f|_{g(t)} \leq C$$

on $M^n \times [0,T)$.

Let $\phi(x,t) = \exp(Bt + \alpha f(x))$, for some suitable B and α. We compute

$$\begin{aligned}\left(\frac{\partial}{\partial t} - \Delta\right) \phi &= \phi \left(B + \alpha \Delta f + \alpha^2 |\nabla f|^2\right) \\ &\geq \phi \left(B - \alpha \sqrt{n} C + \alpha^2 C^2\right).\end{aligned}$$

First choose α big enough so that $\phi(x,t)$ has the desired lower bound as claimed. Then we choose B large enough so that $B - \alpha\sqrt{n}C + \alpha^2 C^2 \geq A$ to obtain (2.9). $\square$

Using ϕ as a barrier we may obtain the following.

COROLLARY 2.23. *If $(M^n, g(t))$ is a complete solution of the Ricci flow with bounded curvature and if $\varphi \geq 0$ satisfies*

$$\frac{\partial}{\partial t}\varphi \leq \Delta\varphi + C\varphi$$
$$\varphi(0) = 0$$

and if $\varphi(x,t) \leq e^{A(d(x,p)+1)}$ for some $A < \infty$, then $\varphi(t) \equiv 0$ for all $t > 0$.

We should point out that, by modifying the proof of Theorem 1.2 in [**404**], one in fact has the following more general version of the maximum principle.

THEOREM 2.24. *Assume that the curvatures of $(M^n, g(t))$, $t \in [0,T)$, are uniformly bounded. If u is a weak subsolution of the heat equation on $M^n \times [0,T]$ with $u(\cdot,0) \leq 0$ and if*

$$\int_0^T \int_{M^n} \exp\left(-\alpha d_{g(0)}^2(x,O)\right) u_+^2(x,t)\, d\mu_{g(t)}(x)\, dt < \infty$$

for some $\alpha > 0$, then $u \leq 0$ on $M^n \times [0,T]$.

Again notice that we can replace $d_{g(0)}(x,O)$ by $d_{g(t)}(x,O)$ due to the fact that they are equivalent under our uniform curvature bound assumption.

LEMMA 2.25. *Let $(M^n, g(t))$, $t \in [0,T)$, be a complete solution to the Ricci flow with bounded curvature and let α_0 be a (p,q)-tensor with*

$$|\alpha_0(x)|_{g(0)} \leq e^{A(d(x,p)+1)}$$

for some $A < \infty$. Let $E^{p,q} \doteqdot (\otimes^p T^* M) \otimes (\otimes^q TM)$ and suppose that

$$F_t : E^{p,q} \to E^{p,q}$$

is a fiber-wise linear map with

$$\|F_t\|_\infty \doteqdot \sup_{\beta(x)\in E^{p,q}} \frac{|F(\beta(x))|_{g(t)}}{|\beta(x)|_{g(t)}} < \infty.$$

Then there exists $B < \infty$ and a solution $\alpha(t)$, $t \in [0,T)$, of

$$\frac{\partial}{\partial t}\alpha = \Delta_{g(t)}\alpha + F_t(\alpha)$$

with $\alpha(0) = \alpha_0$ and $|\alpha|_{g(t)} \leq e^{B(d(x,p)+1)}$. This solution is unique among all solutions with $|\alpha|_{g(t)} \leq e^{C(d(x,p)+1)}$ for all $C < \infty$.

PROOF. It is not difficult to adapt the proof of Proposition in [**402**] to our case. Note that to prove uniqueness we may apply Corollary 2.23 to:

$$\begin{aligned}\frac{\partial}{\partial t}|\alpha|^2 &\leq \Delta|\alpha|^2 - 2|\nabla\alpha|^2 + 2\langle F(\alpha),\alpha\rangle + 2(p+q)|\mathrm{Rc}||\alpha|^2 \\ &\leq \Delta|\alpha|^2 + C|\alpha|^2.\end{aligned}$$

□

COROLLARY 2.26. *Let $(M^n, g(t))$, $t \in [0,T)$, be a complete solution to the Ricci flow with bounded curvature. Under the above hypotheses on the initial data we have the existence and uniqueness of solutions with the corresponding growth conditions for the following equations:*

$$\frac{\partial}{\partial t}X^i = \Delta X^i + R^i_k X^k \tag{2.10}$$

and

$$\frac{\partial}{\partial t}h = \Delta_L h.$$

6. The Einstein-Hilbert functional

The game of Ricci flow, so to speak, is to **control geometric quantities associated to the metric as it evolves**. The above corollary is a nice example of this. Let's now move onto the volume form $d\mu$ (we assume that M is **oriented**). In general, if $\frac{\partial}{\partial s}g_{ij} = v_{ij}$, then

$$\frac{\partial}{\partial s}d\mu = \frac{1}{2}V\,d\mu. \tag{2.11}$$

This is easily seen from the local coordinate formula

$$d\mu = \sqrt{\det g_{ij}}\,dx^1 \wedge \cdots \wedge dx^n \tag{2.12}$$

in a positively oriented local coordinate system $\{x^i\}$ and the formula for the evolution of a determinant of a matrix $A(s)$

$$\frac{d}{ds}\log\det A = \left(A^{-1}\right)^{ij}\frac{d}{ds}A_{ij}, \tag{2.13}$$

where log denotes the natural logarithm. We can see this from the standard definition

$$\det A = \sum_\sigma \mathrm{sign}(\sigma)\,A_{1\sigma(1)}\cdots A_{n\sigma(n)}$$

where the summation is over all permutations σ of $\{1,2,\ldots,n\}$. Differentiating this formula, we get

$$\frac{d}{ds}\det A = \sum_{i,j=1}^n \frac{d}{ds}A_{ij}\sum_{\sigma:\sigma(i)=j}\mathrm{sign}(\sigma)\,A_{1\sigma(1)}\cdots\widehat{A_{i\sigma(i)}}\cdots A_{n\sigma(n)}$$

where $\widehat{A_{i\sigma(i)}}$ means to omit this factor and the second summation is over all permutations σ such that $\sigma(i) = j$. Equation (2.13) now follows from Cramer's rule:

$$\left(A^{-1}\right)_{ij} = \frac{1}{\det A} \sum_{\sigma:\sigma(i)=j} \operatorname{sign}(\sigma) A_{1\sigma(1)} \cdots \widehat{A_{i\sigma(i)}} \cdots A_{n\sigma(n)}.$$

EXERCISE 2.27 (Variation of the inverse of g). *By differentiating the formula* $g^{ij}g_{jk} = \delta^i_k$, *show that*

$$\frac{\partial}{\partial s} g^{ij} = -g^{ik}g^{j\ell}\frac{\partial}{\partial s} g_{k\ell}. \tag{2.14}$$

Of course, the global formula (2.11) is independent of the coordinates we choose to derive the formula. The above formulas give a quick derivation of the first variation formula for the Einstein-Hilbert (total scalar curvature) functional

$$E(g) \doteqdot \int_M R d\mu.$$

Namely, if $\frac{\partial}{\partial s} g_{ij} = v_{ij}$, then

$$\begin{aligned}\frac{d}{ds}E &= \int_M \left(-\Delta V + \nabla_p \nabla_q v_{pq} - \langle v, \mathrm{Rc}\rangle + \frac{1}{2}RV\right) d\mu \\ &= \int_M \left\langle v, \frac{1}{2}Rg - \mathrm{Rc}\right\rangle d\mu.\end{aligned}$$

Note that (twice) the **gradient flow** of E is

$$\frac{\partial}{\partial s} g_{ij} = 2\left(\nabla E(g)\right)_{ij} = Rg_{ij} - 2R_{ij}. \tag{2.15}$$

This looks sort of like Ricci flow, but this equation in fact is not parabolic, and as such, short time existence is not expected to hold. Dropping the Rg term on the RHS of (2.15) yields the Ricci flow.

EXERCISE 2.28. *Given a metric* g_0, *let*

$$\mathcal{C} \doteqdot \{ug_0 : u > 0 \text{ and } \operatorname{Vol}(ug_0) = 1\}$$

be the space of unit volume metrics conformal to g_0. *Show that subject to the constraint of lying in* $\mathcal{C}$, *the critical points* E *have constant scalar curvature.*

7. Evolution of geometric quantities - local coordinate calculations

We now proceed to discuss the variation of the Ricci tensor. Before we can do this, we need to recall the **variation of the Christoffel symbols**.

LEMMA 2.29 (Variation of Christoffel symbols). *If* $g(s)$ *is a one-parameter family of metrics with* $\frac{\partial}{\partial s} g_{ij} = v_{ij}$, *then*

$$\frac{\partial}{\partial s}\Gamma^k_{ij} = \frac{1}{2}g^{k\ell}\left(\nabla_i v_{j\ell} + \nabla_j v_{i\ell} - \nabla_\ell v_{ij}\right). \tag{2.16}$$

PROOF. The derivation of this formula illustrates a nice trick in computing evolutions of various tensor quantities such as the connection and the curvatures. We compute at an arbitrarily chosen point $p \in M$ in normal coordinates centered at p so that $\Gamma_{ij}^k(p) = 0$. Note that $\frac{\partial}{\partial x^i} g_{jk}(p) = 0$. In such coordinates, $\nabla_k a_{i_1 \cdots i_r}^{j_1 \cdots j_q}(p) = \frac{\partial}{\partial x^k} a_{i_1 \cdots i_r}^{j_1 \cdots j_q}(p)$ for any (r, q)-tensor a. Thus, at p we have

$$\frac{\partial}{\partial s} \Gamma_{ij}^k = \frac{1}{2} g^{k\ell} \left(\frac{\partial}{\partial x^i} \frac{\partial}{\partial s} g_{j\ell} + \frac{\partial}{\partial x^j} \frac{\partial}{\partial s} g_{i\ell} - \frac{\partial}{\partial x^\ell} \frac{\partial}{\partial s} g_{ij} \right)$$

and (2.16) follows since $\nabla_i v_{j\ell}(p) = \frac{\partial}{\partial x^i} v_{j\ell}(p)$. Finally we note that since both sides of (2.16) are the components of tensors, equation (2.16) in fact holds as a tensor equation, that is, it is true for any coordinate system, not just normal coordinates. □

REMARK 2.30. *In coordinate free notation, (2.16) is*

(2.17)
$$\left\langle \left(\frac{\partial}{\partial s} \nabla \right) (X, Y), Z \right\rangle = \frac{1}{2} \left((\nabla_X v)(Y, Z) + (\nabla_Y v)(X, Z) - (\nabla_Z v)(X, Y) \right).$$

This formula may derived directly from differentiating (1.3).

COROLLARY 2.31 (Evolution of Christoffel symbols under RF). *Under the Ricci flow $\frac{\partial}{\partial t} g_{ij} = -2R_{ij}$, we have*

$$\boxed{\frac{\partial}{\partial t} \Gamma_{ij}^k = -g^{k\ell} \left(\nabla_i R_{j\ell} + \nabla_j R_{i\ell} - \nabla_\ell R_{ij} \right).} \tag{2.18}$$

A nice consequence of this is the evolution of the Laplacian operator acting on functions.

LEMMA 2.32 (Evolution of laplacian under RF). *If $(M^n, g(t))$ is a solution to the Ricci flow, then*

$$\frac{\partial}{\partial t} \left(\Delta_{g(t)} \right) = 2R_{ij} \cdot \nabla_i \nabla_j,$$

where $\Delta_{g(t)}$ is the Laplacian acting on functions. In particular, when $n = 2$, $\frac{\partial}{\partial t}(\Delta) = R\Delta$.

PROOF. We compute

$$\frac{\partial}{\partial t} \left(\Delta_{g(t)} \right) = \frac{\partial}{\partial t} \left(g^{ij} \nabla_i \nabla_j \right) = -\frac{\partial}{\partial t} g_{ij} \cdot \nabla_i \nabla_j - g^{ij} \left(\frac{\partial}{\partial t} \Gamma_{ij}^k \right) \nabla_k$$

and the result follows from

$$g^{ij} \left(\frac{\partial}{\partial t} \Gamma_{ij}^k \right) = -g^{k\ell} \left(2g^{ij} \nabla_i R_{j\ell} - \nabla_\ell R \right) = 0, \tag{2.19}$$

where we used the contracted second Bianchi identity. □

EXERCISE 2.33. *Given $\frac{\partial}{\partial s} g_{ij} = v_{ij}$, compute $\frac{\partial}{\partial s} \left(\Delta_{g(s)} \right)$.*

SOLUTION.

$$\frac{\partial}{\partial s}\left(\Delta_{g(s)}\right) = -v_{ij}\cdot\nabla_i\nabla_j - \frac{1}{2}g^{ij}g^{k\ell}\left(\nabla_i v_{j\ell} + \nabla_j v_{i\ell} - \nabla_\ell v_{ij}\right)\nabla_k. \tag{2.20}$$

In particular if $v_{ij} = \varphi g_{ij}$ for some function φ, then

$$\frac{\partial}{\partial s}\left(\Delta_{g(s)}\right) = -\varphi\Delta + \frac{n-2}{2}\nabla\varphi\cdot\nabla. \tag{2.21}$$

Now we recall how to get the components of the curvature tensors from the Christoffel symbols. Recall that the components of the Riemann curvature $(3,1)$-tensor defined by $R\left(\frac{\partial}{\partial x^i},\frac{\partial}{\partial x^j}\right)\frac{\partial}{\partial x^k} = R_{ijk}^\ell\frac{\partial}{\partial x^\ell}$ are given by (1.11):

$$R_{ijk}^\ell = \partial_i\Gamma_{jk}^\ell - \partial_j\Gamma_{ik}^\ell + \Gamma_{jk}^p\Gamma_{ip}^\ell - \Gamma_{ik}^p\Gamma_{jp}^\ell \tag{2.22}$$

and the Ricci tensor is $R_{ij} = R_{pij}^p$. From this we calculate the **variation of Ricci in terms of the variation of the connection**

$$\frac{\partial}{\partial s}R_{ij} = \nabla_p\left(\frac{\partial}{\partial s}\Gamma_{ij}^p\right) - \nabla_i\left(\frac{\partial}{\partial s}\Gamma_{pj}^p\right). \tag{2.23}$$

Just as in the proof of Lemma 2.29, this follows from computing at the center in normal coordinates. This is a nice formula and we shall use this later again. For now we just substitute (2.16) into this to obtain that if $\frac{\partial}{\partial s}g_{ij} = v_{ij}$, then

$$\frac{\partial}{\partial s}R_{ij} = \frac{1}{2}\nabla_\ell\left(\nabla_i v_{j\ell} + \nabla_j v_{i\ell} - \nabla_\ell v_{ij}\right) - \frac{1}{2}\nabla_i\nabla_j V. \tag{2.24}$$

Recall $(\operatorname{div} v)_k \doteqdot g^{ij}\nabla_i v_{jk}$. Taking the trace, we obtain the variation formula (2.4) for R

$$\begin{aligned}\frac{\partial}{\partial s}R &= g^{ij}\left(\frac{\partial}{\partial s}R_{ij}\right) - \frac{\partial}{\partial s}g_{ij}\cdot R_{ij}\\ &= \nabla_\ell\nabla_i v_{i\ell} - \Delta V - v_{ij}\cdot R_{ij}.\end{aligned} \tag{2.25}$$

Commuting derivatives in (2.24) yields the **variation of Ricci formula**:

$$\boxed{\frac{\partial}{\partial s}R_{ij} = -\tfrac{1}{2}\left(\Delta_L v_{ij} + \nabla_i\nabla_j V - \nabla_i\left(\operatorname{div} v\right)_j - \nabla_j\left(\operatorname{div} v\right)_i\right).} \tag{2.26}$$

Here Δ_L denotes the **Lichnerowicz Laplacian**, which is defined by

$$\boxed{\Delta_L v_{ij} \doteqdot \Delta v_{ij} + 2R_{kij\ell}v_{k\ell} - R_{ik}v_{jk} - R_{jk}v_{ik}} \tag{2.27}$$

acting on symmetric 2-tensors. Recall that $R_{kij\ell} \doteqdot g_{\ell p} R_{kij}^{p}$. We obtain (2.26) from (2.24) by the computation:

$$\begin{aligned}\nabla_\ell \left(\nabla_i v_{j\ell} + \nabla_j v_{i\ell}\right) &= \nabla_i \nabla_\ell v_{j\ell} - R_{\ell ijm} v_{m\ell} - R_{\ell i\ell m} v_{jm} \\ &+ \nabla_j \nabla_\ell v_{i\ell} - R_{\ell jim} v_{m\ell} - R_{\ell j\ell m} v_{im} \\ &= \nabla_i \left(\operatorname{div} v\right)_j + \nabla_j \left(\operatorname{div} v\right)_i - 2R_{\ell ijm} v_{\ell m} \\ &+ R_{im} v_{jm} + R_{jm} v_{im},\end{aligned}$$

where we used the commutator formula (1.30). Note that (2.26) may be rewritten as

$$\frac{\partial}{\partial s}\left(-2R_{ij}\right) = \Delta_L v_{ij} + \nabla_i X_j + \nabla_j X_i$$

where $X = \frac{1}{2}\nabla V - \operatorname{div} v$. This is related to DeTurck's trick in proving short time existence (see (2.40).)

The Lichnerowicz Laplacian is a fundamental operator; when acting on symmetric 2-tensors in the context of Ricci flow it is perhaps more natural than the rough Laplacian $\Delta = g^{ij}\nabla_i\nabla_j$. Examples of this naturality are the appearance of Δ_L in the linearized Ricci flow equation (formula (2.37) below is an example of this) and the following identity, which we will use in Chapter **??**.

LEMMA 2.34 (Hessian and Lichnerowicz heat operator commutator formula). *Under the Ricci flow, the Hessian and the Lichnerowicz Laplacian heat operator commute. That is, for any function f of space and time we have*

$$\boxed{\nabla_i \nabla_j \left(\frac{\partial f}{\partial t} - \Delta f\right) = \left(\frac{\partial}{\partial t} - \Delta_L\right) \nabla_i \nabla_j f.} \tag{2.28}$$

PROOF. Using (1.30) we compute

$$\begin{aligned}\nabla_i \nabla_j \nabla_k \nabla_k f &= \nabla_i \nabla_k \nabla_k \nabla_j f - \nabla_i \left(R_{j\ell} \nabla_\ell f\right) \\ &= \nabla_k \nabla_i \nabla_k \nabla_j f - R_{ikj\ell} \nabla_\ell \nabla_k f - R_{i\ell} \nabla_j \nabla_\ell f \\ &- \nabla_i R_{j\ell} \nabla_\ell f - R_{j\ell} \nabla_i \nabla_\ell f \\ &= \nabla_k \nabla_k \nabla_i \nabla_j f - \nabla_k \left(R_{ikj\ell} \nabla_\ell f\right) - R_{ikj\ell} \nabla_\ell \nabla_k f - R_{i\ell} \nabla_j \nabla_\ell f \\ &- \nabla_i R_{j\ell} \nabla_\ell f - R_{j\ell} \nabla_i \nabla_\ell f \\ &= \Delta \nabla_i \nabla_j f + \left(\nabla_j R_{i\ell} - \nabla_\ell R_{ij} - \nabla_i R_{j\ell}\right) \nabla_\ell f - 2R_{ikj\ell} \nabla_\ell \nabla_k f \\ &- R_{i\ell} \nabla_j \nabla_\ell f - R_{j\ell} \nabla_i \nabla_\ell f \\ &= \Delta_L \nabla_i \nabla_j f + \left(\nabla_j R_{i\ell} - \nabla_\ell R_{ij} - \nabla_i R_{j\ell}\right) \nabla_\ell f\end{aligned}$$

where we used $\nabla_k R_{ikj\ell} = \nabla_j R_{i\ell} - \nabla_\ell R_{ij}$ (from the second Bianchi identity) to get the last equality. Second, using (2.18), we compute

$$\frac{\partial}{\partial t} \nabla_i \nabla_j f = \nabla_i \nabla_j \frac{\partial f}{\partial t} + \left(\nabla_j R_{i\ell} - \nabla_\ell R_{ij} - \nabla_i R_{j\ell}\right) \nabla_\ell f. \tag{2.29}$$

Formula (2.28) now follows from combining the above two calculations. □

COROLLARY 2.35. *If $g(t)$ satisfies the Ricci flow and $f(t)$ satisfies the heat equation $\frac{\partial f}{\partial t} = \Delta f$, then the Hessian satisfies the Lichnerowicz Laplacian heat equation:*

$$\frac{\partial}{\partial t}(\nabla\nabla f) = \Delta_L(\nabla\nabla f).$$

EXERCISE 2.36 (Commutator of $\frac{\partial}{\partial t} + \Delta_L$ and $\nabla\nabla$). *Using the formulas derived in the proof of Lemma 2.34, establish under Ricci flow we have the identity*

$$\nabla_i\nabla_j\left(\frac{\partial f}{\partial t} + \Delta f\right) = \left(\frac{\partial}{\partial t} + \Delta_L\right)\nabla_i\nabla_j f - 2\left(\nabla_i R_{j\ell} + \nabla_j R_{i\ell} - \nabla_\ell R_{ij}\right)\nabla_\ell f.$$

Recall the exterior derivative of a p-form ω may be expressed in terms of covariant derivatives as

$$(d\omega)_{i_0 i_1 \cdots i_p} = \sum_{j=0}^{p}(-1)^j \nabla_{i_j}\omega_{i_0 i_1 \cdots \widehat{i_j} \cdots i_p}.$$

The adjoint δ is given by

$$(\delta\alpha)_{i_1 \cdots i_{p-1}} = -g^{jk}\nabla_j \alpha_{k i_1 \cdots i_{p-1}}.$$

Indeed, one easily verifies that for any $(p-1)$-form β and p-form α

$$\int_{M^n}\langle d\beta, \alpha\rangle\, d\mu = \int_{M^n}\langle \beta, \delta\alpha\rangle\, d\mu.$$

The **Hodge Laplacian** acting on p-forms is defined by $\Delta_d \doteqdot -(d\delta + \delta d)$ (we have adopted the opposite of the usual sign convention). If β is a 2-form, then

$$(\Delta_d\beta)_{ij} = \Delta\beta_{ij} + 2R_{ik\ell j}\beta_{k\ell} - R_{ik}\beta_{kj} - R_{jk}\beta_{ik}. \tag{2.30}$$

So the Hodge Laplacian acts on 2-forms formally in the same way as the Lichnerowicz Laplacian acts on symmetric 2-tensors.

EXERCISE 2.37 (Bochner [**51**], [**52**]). *Show that if X is a 1-form, then*

$$\Delta X_i - R_{ij}X_j = \Delta_d X_i. \tag{2.31}$$

In particular, if the Ricci curvature of a closed manifold is positive, then there are no nontrivial harmonic 1-forms. By the Hodge theorem, this implies that the first Betti number $b_1(M)$ is zero. This is a consequence of Myers' Theorem that the fundamental group of M is finite.

SOLUTION. Recall $\Delta_d = -(d\delta + \delta d)$. We compute

$$(dX)_{ij} = \nabla_i X_j - \nabla_j X_i$$
$$(\delta dX)_j = -\nabla_i (dX)_{ij} = -\nabla_i(\nabla_i X_j - \nabla_j X_i)$$

and

$$\delta X = -\nabla_i X_i$$
$$(d\delta X)_j = -\nabla_j\nabla_i X_i$$

so that

$$\begin{aligned}(\Delta_d X)_i &= \nabla_i (\nabla_i X_j - \nabla_j X_i) + \nabla_j \nabla_i X_i \\ &= (\Delta X)_j + (\nabla_j \nabla_i - \nabla_i \nabla_j) X_i \\ &= (\Delta X)_j - R_{jk} X_k.\end{aligned}$$

EXERCISE 2.38. *Verify (2.30).* HINT: *using*

$$(d\beta)_{ijk} = \nabla_i \beta_{jk} - \nabla_j \beta_{ik} + \nabla_k \beta_{ij} \qquad (\delta\beta)_k = -\nabla_i \alpha_{ik},$$

show that

$$(\Delta_d \beta)_{jk} = \Delta \beta_{jk} + (\nabla_j \nabla_i - \nabla_i \nabla_j) \beta_{ik} + (\nabla_i \nabla_k - \nabla_k \nabla_i) \beta_{ij}$$

and apply the commutator formulas for covariant differentiation.

EXERCISE 2.39. *Show that under the Ricci flow, for any 1-form X*

$$\left(\frac{\partial}{\partial t} - \Delta_L\right)(\mathcal{L}_{X^\natural} g) = \mathcal{L}_{\left[\left(\frac{\partial}{\partial t} - \Delta_d\right) X\right]^\natural} g \ ;$$

that is,

$$\left(\frac{\partial}{\partial t} - \Delta_L\right)(\nabla_i X_j + \nabla_j X_i) = \nabla_i Y_j + \nabla_j Y_i, \tag{2.32}$$

where $Y \doteqdot \left(\frac{\partial}{\partial t} - \Delta_d\right) X$. Note that by taking $X = df$, we obtain (2.28) since $\left(\frac{\partial}{\partial t} - \Delta_d\right) df = d\left(\frac{\partial}{\partial t} - \Delta\right) f$.

SOLUTION. From

$$\frac{\partial}{\partial t}(\nabla_i X_j) = \nabla_i \left(\frac{\partial}{\partial t} X_j\right) + (\nabla_i R_{jk} + \nabla_j R_{ik} - \nabla_k R_{ij}) X_k$$

and

$$\begin{aligned}\nabla_i (\Delta X_j - R_{jk} X_k) = {}& \Delta (\nabla_i X_j) + 2R_{kij\ell} \nabla_k X_\ell - R_{ik} \nabla_k X_j - R_{jk} \nabla_i X_k \\ & - (\nabla_i R_{jk} + \nabla_j R_{ik} - \nabla_k R_{ij}) X_k\end{aligned}$$

we conclude

$$\begin{aligned}&\nabla_i \left(\frac{\partial}{\partial t} X_j - (\Delta X_j - R_{jk} X_k)\right) + \nabla_j \left(\frac{\partial}{\partial t} X_i - (\Delta X_i - R_{ik} X_k)\right) \\ &= \left(\frac{\partial}{\partial t} - \Delta_L\right)(\nabla_i X_j + \nabla_j X_i).\end{aligned}$$

A useful consequence of the above exercise is the following.

LEMMA 2.40. *If $(M^n, g(t))$ is a solution to the Ricci flow and if X is a vector field evolving by*

$$\frac{\partial}{\partial t} X^i = \Delta X^i + R^i_k X^k, \tag{2.33}$$

then $h_{ij} \doteqdot \nabla_i X_j + \nabla_j X_i = (\mathcal{L}_X g)_{ij}$ evolves by

$$\frac{\partial}{\partial t} h_{ij} = \Delta_L h_{ij} \doteqdot \Delta h_{ij} + 2R_{kij\ell} h_{k\ell} - R_{ik} h_{kj} - R_{jk} h_{ik}. \tag{2.34}$$

PROOF. The dual 1-form evolves by

$$\frac{\partial}{\partial t} X_i = \Delta X_i - R_{ij} X_j = \Delta_d X_i$$

where $\Delta_d \doteqdot -(d\delta + \delta d)$. The result now follows from Exercise 2.39. □

REMARK 2.41. *A special case of (2.32) is formula (2.28), which implies that if* $\frac{\partial f}{\partial t} = \Delta f$, *then* $\frac{\partial}{\partial t}(df) = \Delta_d (df)$ *and*

$$\frac{\partial}{\partial t}(\nabla_i \nabla_j f) = -(\Delta_L \nabla\nabla f)_{ij}.$$

EXERCISE 2.42. *Show that if* X *is a Killing vector field, then*

$$\nabla_k \nabla_j X_i + R_{\ell kji} X_\ell = 0. \tag{2.35}$$

SOLUTION ([**231**], p. 108). We compute

$$\begin{aligned}
0 &= \nabla_k (\nabla_j X_i + \nabla_i X_j) + \nabla_j (\nabla_i X_k + \nabla_k X_i) + \nabla_i (\nabla_j X_k + \nabla_k X_j) \\
&= (\nabla_k \nabla_j X_i + \nabla_i \nabla_k X_j) + (\nabla_j \nabla_i X_k + \nabla_k \nabla_i X_j) + (\nabla_j \nabla_k X_i + \nabla_i \nabla_j X_k) \\
&= (\nabla_i \nabla_k X_j - \nabla_k \nabla_i X_j) - (\nabla_j \nabla_k X_i + \nabla_k \nabla_j X_i) + (\nabla_i \nabla_j X_k - \nabla_j \nabla_i X_k) \\
&= R_{kij\ell} X_\ell - 2\nabla_k \nabla_j X_i + R_{jki\ell} X_\ell + R_{jik\ell} X_\ell \\
&= -2\nabla_k \nabla_j X_i + 2R_{jik\ell} X_\ell
\end{aligned}$$

where we used the first Bianchi identity to get the last equality.

Tracing (2.35) we have

$$\Delta X_i + R_{\ell i} X_\ell = 0. \tag{2.36}$$

Hence, if M^n is closed and the Ricci curvature is negative, then there are no nontrivial Killing vector fields. For a generalization of this to conformal Killing vector vector fields, see Proposition 6.4.

Since, by the contracted second Bianchi identity,

$$\nabla_i \nabla_j R - \nabla_i (\operatorname{div} \mathrm{Rc})_j - \nabla_j (\operatorname{div} \mathrm{Rc})_i = 0,$$

equation (2.26) implies the following.

LEMMA 2.43 (Evolution of the Ricci tensor under RF). *Under the Ricci flow,*

$$\boxed{\frac{\partial}{\partial t} R_{ij} = \Delta_L R_{ij} = \Delta R_{ij} + 2R_{kij\ell} R_{k\ell} - 2R_{ik} R_{jk}.} \tag{2.37}$$

Now, just like the evolution equation for the scalar curvature, we have a heat-type equation. However, there is an important difference: R is a scalar function whereas R_{ij} is a tensor. It is nice to know that we can still apply the maximum principle (see [**153**], Theorem 4.6 on p. 97). This (the maximum principle for tensors) is the subject of the first section of the next chapter.

EXERCISE 2.44. *Calculate the evolution equation for* $R_{ij} - \alpha R g_{ij}$, *where* $\alpha \in \mathbb{R}$.

EXERCISE 2.45. *Using (2.22) show that*

$$R_{ijk\ell} = \frac{1}{2}\left(\partial_i\partial_k h_{j\ell} - \partial_i\partial_\ell h_{jk} - \partial_j\partial_k h_{i\ell} + \partial_j\partial_\ell h_{ik}\right) + Q\left(g, \partial g\right)$$

where Q is quadratic in ∂g. (For a related formula, see (3.14).)

EXERCISE 2.46. *Show that in normal coordinates*

$$\partial_i\partial_j g_{k\ell} = \frac{1}{6}\left(R_{ik\ell j} + R_{i\ell jk}\right).$$

8. DeTurck's trick and short time existence

Using (2.26) we can now give a rough description of how DeTurck's proof (or **DeTurck's trick**) of short time existence works. First note that the principal symbol of the nonlinear partial differential operator $-2\operatorname{Rc}(g)$ of the metric g is nonnegative and has a nontrivial kernel which is due exactly to the diffeomorphism invariance of the Ricci tensor (see [**153**], §2.3 for details). For this reason the Ricci flow equation is only weakly parabolic. We search for an equivalent flow which is strictly parabolic. Motivated by formula (2.26), given a fixed background connection $\tilde{\Gamma}$, which for convenience we assume to be the Levi-Civita connection of a metric $\tilde{g}$, we define the **Ricci–DeTurck flow** by

$$\begin{aligned} \frac{\partial}{\partial t} g_{ij} &= -2R_{ij} + \nabla_i W_j + \nabla_j W_i\,, \\ g(0) &= g_0, \end{aligned} \tag{2.38}$$

where the time-dependent 1-form $W = W(g)$ is defined by

$$W_j \doteqdot g_{jk} g^{pq}\left(\Gamma_{pq}^k - \tilde{\Gamma}_{pq}^k\right). \tag{2.39}$$

Note that if $g(s)$ is a one-parameter family of metrics with $g(s) = g$ and

$$\left.\frac{\partial}{\partial s}\right|_{s=0} g_{ij} = v_{ij},$$

then

$$\left.\frac{\partial}{\partial s}\right|_{s=0} W\left(g(s)\right)_j = -X_j + \text{zeroth order terms in } v.$$

where $X = \frac{1}{2}\nabla V - \operatorname{div} v$ as above. We compute

$$\left.\frac{\partial}{\partial s}\right|_{s=0}\left(-2R_{ij} + \nabla_i W_j + \nabla_j W_i\right) = \Delta_L v_{ij} + \text{first order terms in } v. \tag{2.40}$$

EXERCISE 2.47. *Verify formula (2.40) by using (2.26), (2.16).*

From (2.40) it follows that the Ricci-DeTurck flow is strictly parabolic and that given any smooth initial metric g_0 on a closed manifold, there exists a unique solution $g(t)$ to the Ricci-DeTurck flow with $g(0) = g_0$. Note we may also rewrite the Ricci-DeTurck flow (2.38) as (see also Volume 2)

$$\frac{\partial}{\partial t} g_{ij} = g^{\ell m}\tilde{\nabla}_\ell\tilde{\nabla}_m g_{ij} + g^{-1} * g * \tilde{g}^{-1} * \widetilde{\operatorname{Rm}} + g * g^{-3} * \left(\tilde{\nabla} g\right)^2,$$

which also exhibits the strict parabolicity of the flow. Now given a solution of the Ricci-DeTurck flow, we can solve the following ODE at each point in M :

$$\begin{aligned}\frac{\partial}{\partial t}\varphi_t &= -W^* \\ \varphi_0 &= \mathrm{id},\end{aligned} \tag{2.41}$$

where $W^*(t)$ is the vector field dual to $W(t)$ with respect to $g(t)$. Pulling back $g(t)$ by the diffeomorphisms φ_t, we obtain a solution

$$\bar{g}(t) \doteqdot (\varphi_t)^* g(t) \tag{2.42}$$

to the Ricci flow with $\bar{g}(0) = g_0$ (see p.81 of [**153**] for instance). One can also show that this solution is unique (p. 90 of [**153**]).

THEOREM 2.48 (Hamilton, DeTurck - Short time existence). *If M^n is a closed Riemannian manifold and if g_0 is a C^∞ Riemannian metric, then there exists a unique smooth solution $\bar{g}(t)$ to the Ricci flow define on some time interval $[0,\delta)$, $\delta > 0$, with $\bar{g}(0) = g_0$.*

Recall that given a map $f : (M^n, g) \to (N^m, h)$, the **map Laplacian** of f is defined by

$$\begin{aligned}(\Delta_{g,h} f)^\gamma &= \Delta_g(f^\gamma) + g^{ij}\left(\Gamma(h)^\gamma_{\alpha\beta} \circ f\right)\frac{\partial f^\alpha}{\partial x^i}\frac{\partial f^\beta}{\partial x^j} \\ &= g^{ij}\left(\frac{\partial^2 f^\gamma}{\partial x^i \partial x^j} - \Gamma^k_{ij}\frac{\partial f^\gamma}{\partial x^k} + \left(\Gamma(h)^\gamma_{\alpha\beta} \circ f\right)\frac{\partial f^\alpha}{\partial x^i}\frac{\partial f^\beta}{\partial x^j}\right),\end{aligned} \tag{2.43}$$

where $f^\gamma \doteqdot y^\gamma \circ f$, and $\{x^i\}$ and $\{y^\alpha\}$ are coordinates on M and N, respectively. Note that $\Delta_{g,h} f \in C^\infty(f^* TN)$, where $f^*(TN) \to M$ is the pullback vector bundle of TN by f. In (2.43) $\Delta_g(f^\gamma)$ denotes the Laplacian with respect to g of the function f^γ. As a special case, if $M = N$ and f is the identity map and we choose the x and y coordinates to be the same, then

$$(\Delta_{g,h}\,\mathrm{id})^k = g^{ij}\left(-\Gamma(g)^k_{ij} + \Gamma(h)^k_{ij}\right).$$

REMARK 2.49. *The derivative df of a map $f : M^n \to N^m$ is a section of the vector bundle $E \doteqdot T^*M \otimes f^*TN$. On E is a natural metric and compatible connection $\nabla^{g,h}$ defined by the (dual of the) Riemannian metric g and associated Levi-Civita connection on T^*M and the pullback by f of the metric h and its associated Levi-Civita connection on TN. So $\nabla^{g,h} df$ is a section of the bundle $T^*M \otimes_S T^*M \otimes f^*TN$. The map Laplacian is the trace with respect to g of $\nabla^{g,h} df$*

$$\Delta_{g,h} f = \operatorname{tr}_g\left(\nabla^{g,h} df\right).$$

A map $f : (M^n, g) \to (N^m, h)$ is called a **harmonic map** if $\Delta_{g,h} f = 0$. In the case where $N = \mathbb{R}$ a harmonic map is the same as a harmonic function. If M is 1-dimensional, then a harmonic map is the same as a constant speed geodesic.

REMARK 2.50. *Given a diffeomorphism $f : (M^n, g) \to (N^n, h)$, the map Laplacian satisfies the following identity:*

$$(\Delta_{g,h} f)(x) = \left(\Delta_{(f^{-1})^* g, h} \operatorname{id}_N\right)(f(x)) \in C^\infty(f^* TN) \tag{2.44}$$

This corresponds to considering f as

$$(M^n, g) \xrightarrow{f} \left(N^n, \left(f^{-1}\right)^* g\right) \xrightarrow{\operatorname{id}} (N^n, h)$$

where the map on the left f is an isometry. More generally, if (P^n, k) and (N^m, h) are Riemannian manifolds, $F : P^n \to N^m$ is a map and $\varphi : M^n \to P^n$ is a diffeomorphism, then

$$(\Delta_{k,h} F)(\varphi(y)) = (\Delta_{\varphi^* k, h}(F \circ \varphi))(y). \tag{2.45}$$

which corresponds to

$$(M^n, \varphi^* k) \xrightarrow{\varphi} (P^n, k) \xrightarrow{F} (N^m, h).$$

Formula (2.44) is the special case of (2.45) where $n = m$, $P^n = N^m$, $F = \operatorname{id}_N$, $\varphi = f$ and $k = \left(f^{-1}\right)^* g$. To prove (2.45) we compute

$$\begin{aligned}
(\Delta_{\varphi^* k, h}(F \circ \varphi))^\ell &= \Delta_{\varphi^* k}\left(x^\ell \circ F \circ \varphi\right) + (\varphi^* k)^{\alpha\beta}\left(\Gamma(h)_{ij}^\ell \circ F \circ \varphi\right) \frac{\partial (F \circ \varphi)^i}{\partial y^\alpha} \frac{\partial (F \circ \varphi)^j}{\partial y^\beta} \\
&= \left(\Delta_k\left(x^\ell \circ F\right)\right) \circ \varphi + \left(k^{ab} \circ \varphi\right)\left(\Gamma(h)_{ij}^\ell \circ F \circ \varphi\right)\left(\frac{\partial F^i}{\partial z^a} \circ \varphi\right)\left(\frac{\partial F^j}{\partial z^b} \circ \varphi\right) \\
&= (\Delta_{k,h} F)^\ell \circ \varphi.
\end{aligned}$$

where $\varphi^i = \varphi \circ x^i$, k^{ab} is the inverse of $k_{ab} \doteqdot k\left(\frac{\partial}{\partial z^a} \frac{\partial}{\partial z^b}\right)$ and since $\Delta_{\varphi^* g}(u \circ \varphi) = \Delta_g u$ for a function u.

If $\tilde{\Gamma}$ is the Levi-Civita connection of a metric $\tilde{g}$, then equation (2.41) is equivalent to

$$\frac{\partial}{\partial t} \varphi_t = g^{pq}\left(-\Gamma_{pq}^k + \tilde{\Gamma}_{pq}^k\right) \frac{\partial}{\partial x^k} = \Delta_{g,\tilde{g}} \operatorname{id} = \Delta_{\bar{g}(t),\tilde{g}} \varphi_t.$$

In other words, if $\bar{g}(t)$ is a solution to the Ricci flow and $\varphi_t : M^n \to M^n$ is a solution to the harmonic map heat flow

$$\frac{\partial}{\partial t} \varphi_t = \Delta_{\bar{g}(t),\tilde{g}} \varphi_t,$$

then the metrics $g(t) \doteqdot \varphi_t \bar{g}(t)$ satisfy the Ricci-DeTurck flow. See Chapter 3, §4 of [**153**] for more details.

Finally we make some remarks on the uniqueness problem for complete solutions on noncompact manifolds; the idea is the same as in compact case (see Step 4 on p. 90 of [**153**]). Let $g_1(t)$ and $g_2(t)$ be two complete solutions of the Ricci flow with $g_1(0) = g_2(0) = g_0$ on a noncompact manifold M^n.

Suppose that we can show that there exist solutions $\varphi_1(t), \varphi_2(t) : M^n \to M^n$ to the harmonic map heat flow

$$\frac{\partial}{\partial t}\varphi_i(t) = \Delta_{g_i(t),g_0}\varphi_i(t)$$
$$\varphi_i(0) = \mathrm{id}_M$$

on some short time interval $[0, \varepsilon)$. Then the metrics

$$\hat{g}_i(t) = \varphi_i(t)_* g_i(t) = \left(\varphi_i(t)^{-1}\right)^* g_i(t)$$

are solutions of the Ricci-DeTurck flow with $\hat{g}_1(0) = \hat{g}_2(0) = g_0$. By the uniqueness theorem for the Ricci-DeTurck flow we have

$$\hat{g}_1(t) = \hat{g}_2(t) \doteqdot \hat{g}(t)$$

for $t \in [0, \varepsilon)$. We also have that $\varphi_1(t)$ and $\varphi_2(t)$ are solutions of the ODE (2.41)

$$\frac{d}{dt}(\varphi_i(t)) = -W(t) \circ \varphi_i(t)$$
$$\varphi_i(0) = \mathrm{id}_M$$

where

$$W(t)^k \doteqdot \hat{g}(t)^{pq}\left(\Gamma(\hat{g}(t))_{pq}^k - \Gamma(g_0)_{pq}^k\right).$$

Hence $\varphi_1(t) = \varphi_2(t)$ for $t \in [0, \varepsilon)$. We conclude that $g_1(t) = g_2(t)$ for $t \in [0, \varepsilon)$. Thus the uniqueness problem on noncompact manifolds reduces to the short time existence problem for the harmonic map heat flow with respect to a time-dependent domain metric. For some further developments see [**117**], [**118**] and [**357**].

9. Notes and commentary

§**5**. Without growth assumptions, solutions to the heat equation on $\mathbb{R}^n$ with a given initial data are not unique. A classic example (see Cannon [**70**] or Widder [**513**]) in dimension 1 is given by the solution

$$u(x,t) = \sum_{n=0}^{\infty} f^{(n)}(t)\frac{x^{2n}}{(2n)!} \tag{2.46}$$

where

$$f(t) = \begin{cases} \exp\left\{-t^{-2}\right\} & t \neq 0 \\ 0 & t = 0 \end{cases}.$$

This series converges and can be differentiated term by term to get

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} = \sum_{n=0}^{\infty} f^{(n+1)}(t)\frac{x^{2n}}{(2n)!}.$$

So in fact u is a C^∞ solution to the heat equation on $\mathbb{R} \times (-\infty, \infty)$ which is identically zero at $t = 0$. Note $u(0,t) = f(t) > 0$ for $t \neq 0$.

§**7**. We recall the **deRham Theorem**, some basic formulas, and the **Hodge Decomposition theorem**. The exterior derivative forms a complex

$$0 \to \Omega^0(M^n) \xrightarrow{d} \Omega^1(M^n) \xrightarrow{d} \cdots \xrightarrow{d} \Omega^{n-1}(M^n) \xrightarrow{d} \Omega^n(M^n) \to 0,$$

where $d^2 \doteqdot d \circ d = 0$. Hence $\operatorname{image}(d) \subset \ker(d)$ and we can define the r**th deRham cohomology group**:

$$H^r_{\mathrm{deR}}(M) \doteqdot \frac{\ker(d|_{\Omega^r(M)})}{\operatorname{image}(d|_{\Omega^{r-1}(M)})}.$$

We have

THEOREM 2.51 (deRham). *The rth deRham cohomology group is isomorphic to the rth singular real cohomology group:*

$$H^r_{\mathrm{deR}}(M) \cong H^r(M;\mathbb{R}).$$

Recall that the exterior derivative d acting on r-forms may be expressed in terms of covariant derivatives as follows:

$$d\,\alpha(X_0,\dots,X_r) = \sum_{j=0}^{r}(-1)^j\,\nabla_{X_j}\alpha(X_0,\dots,X_{j-1},X_{j+1},\dots,X_r),$$

or in local coordinates,

$$d\alpha_{i_0\cdots i_r} = \sum_{j=0}^{r}(-1)^j\,\nabla_{i_j}\,\alpha_{i_0\cdots i_{j-1}i_{j+1}\cdots i_r}.$$

The L^2-inner product on the space of r-forms is given by:

$$\langle \alpha,\beta\rangle = \int_M g^{i_1 j_1}\cdots g^{i_r j_r}\,\alpha_{i_1\cdots i_r}.\beta_{j_1\cdots j_r}\,d\mu.$$

The adjoint δ of d with respect to the L^2-inner product is defined by the relation:

$$\langle d\alpha,\beta\rangle \doteqdot \langle \alpha,\delta\beta\rangle,$$

where $\alpha \in \Omega^r(M)$ and $\beta \in \Omega^{r+1}(M)$. In terms of d and the **Hodge star operator** $* : \Omega^p(M) \to \Omega^{n-p}(M)$, it is given by the formula:

$$\delta\beta = (-1)^{nr+1} * d * \beta.$$

In terms of covariant derivatives, it may be written as:

$$\delta\beta(X_1,\dots,X_r) = -\sum_{i=1}^{n}\nabla_{e_i}\beta(e_i,X_1,\dots,X_r),$$

where $\{e_i\}_{i=1}^n$ is an local orthonormal frame field. In local coordinates, we write:

$$\delta\beta_{i_1\cdots i_r} = -g^{k\ell}\nabla_k\,\alpha_{\ell i_1\cdots i_r}.$$

The **Hodge Laplacian** $\triangle_d$ acting on differential forms is defined by:

$$\triangle_d = -(d\delta + \delta d).$$

Acting on functions, it is the same as the Laplace-Beltrami operator. In local coordinates we may write the Hodge Laplacian as:

$$\begin{aligned}(\triangle_d \alpha)_{i_1 \cdots i_r} &= (-1)^{j+1} g^{k\ell} \nabla_{i_j} \nabla_k \alpha_{\ell i_1 \cdots i_{j-1} i_{j+1} \cdots i_r} + g^{k\ell} \nabla_k \nabla_\ell \alpha_{i_1 \cdots i_r} \\ &\quad + (-1)^j g^{k\ell} \nabla_k \nabla_{i_j} \alpha_{\ell i_1 \cdots i_{j-1} i_{j+1} \cdots i_r}.\end{aligned}$$

A differential form α is called **harmonic** if (Kodaira 1949)

$$\triangle_d \alpha = 0.$$

Since

$$\int_M \langle \triangle_d \alpha, \alpha \rangle \, d\mu = -\int_M \left(|d\alpha|^2 + |\delta\alpha|^2 \right) d\mu,$$

we have α is harmonic if and only if (Hodge 1952)

$$d\alpha = 0 \text{ and } \delta\alpha = 0.$$

The space of harmonic r-forms is denoted by:

$$H^r = \{\alpha \in \Omega^r(M) : \triangle_d \alpha = 0\}.$$

The essence of the Hodge theorem is understanding when, given $\gamma \in \Omega^r(M)$, the equation:

$$\triangle_d \alpha = \gamma$$

has a solution $\alpha \in \Omega^r(M)$. If $\beta \in H^r$ is a harmonic r-form, then

$$\langle \gamma, \beta \rangle = \langle \triangle_d \alpha, \beta \rangle = \langle \alpha, \triangle_d \beta \rangle = 0.$$

We have following, which is known as the Hodge Decomposition Theorem.

THEOREM 2.52 (Hodge 1952). *Given $\gamma \in \Omega^r(M)$, the equation:*

$$\triangle_d \alpha = \gamma$$

has a solution $\alpha \in \Omega^r(M)$ if and only if

$$\langle \gamma, \beta \rangle = 0,$$

for all $\beta \in H^r$. Consequently, we have the following decomposition of the space of r-forms:

$$\begin{aligned}\Omega^r(M) &= \triangle_d (\Omega^r(M)) \oplus H^r \\ &= d\delta (\Omega^r(M)) \oplus \delta d (\Omega^r(M)) \oplus H^r.\end{aligned}$$

Moreover, the space H^r is finite-dimensional.

COROLLARY 2.53. *In each deRham cohomology class, there is a unique harmonic form representing the cohomology class. In particular, the rth deRham cohomology group is isomorphic to the space of harmonic r-forms H^r.*

An important property of the Hodge Laplacian is that it commutes with the star operator:

$$\triangle_d * = * \triangle_d .$$

Thus, if $\alpha \in H^r$ is a harmonic r-form, then $*\alpha$ is a harmonic $(n-r)$-form, i.e.,

$$* : H^r \to H^{n-r}$$

is an isomorphism. The corollary then implies:

$$H^r_{\text{deR}}(M) \cong H^{n-r}_{\text{deR}}(M),$$

which is known as the **Poincaré duality theorem** for deRham cohomology (and by the deRham theorem this is also true for singular real cohomology.)

§8. For a discussion of the relation between DeTurck's trick and the harmonic map heat flow, see [**441**] and §6 of [**267**] (there is also an exposition in Chapter 3, §4 of [**153**]). Let

$$\sigma = \sigma D\left(-2\operatorname{Rc}\right)(\zeta) : S^2T^*M^n \to S^2T^*M^n$$

denote the symbol of the linearization of the Ricci tensor as a function of the metric. Assuming that $\zeta_1 = 1$ and $\zeta_i = 0$ for $i \neq 1$, one computes (see [**255**]) that

$$\begin{aligned} \sigma(T)_{ij} &= T_{ij} \qquad \text{if } i,j \neq 1 \\ \sigma(T)_{1j} &= 0 \qquad \text{if } j \neq 1 \\ \sigma(T)_{11} &= \sum_{k=2}^{n} T_{kk}. \end{aligned} \tag{2.47}$$

One checks that σ is given by a nonnegative $N \times N$ matrix, where $N = n(n+1)/2$, and its kernel is the n-dimensional subspace given by

$$\ker \sigma D\left(-2\operatorname{Rc}\right)(\zeta) = \left\{ T : T_{ij} = 0 \text{ for } i,j \neq 1 \text{ and } \sum_{k=2}^{n} T_{kk} = 0 \right\}.$$

This kernel is due exactly to the diffeomorphism invariance of the operator $g \mapsto -2\operatorname{Rc}$ which we see as follows. Define the linear **Bianchi operator**

$$B_g : C^\infty\left(S^2T^*M^n\right) \to C^\infty\left(T^*M^n\right)$$

by

$$B_g(h)_k \doteqdot g^{ij}\left(\nabla_i h_{jk} - \frac{1}{2}\nabla_k h_{ij}\right)$$

so that $B_g(-2\operatorname{Rc}) = 0$. One finds that

$$K \doteqdot \ker \sigma B_g(\zeta) = \operatorname{image} \sigma D\left(-2\operatorname{Rc}\right)(\zeta) \subset S^2T^*M^n$$

are equal to

$$K = \left\{ T : T_{1j} = 0 \text{ for } j \neq 1 \text{ and } T_{11} = \sum_{k=2}^{n} T_{kk} \right\}.$$

From (2.47) we see that

$$\sigma|_K (T) = |\zeta|^2 T$$

for any $\zeta \in T^*M^n$. For comparison with the cross curvature flow, we note that when $n = 3$, $\sigma = \sigma D (-2\,\mathrm{Rc})(\zeta)$ is given by

$$\sigma \begin{pmatrix} T_{11} \\ T_{12} \\ T_{13} \\ T_{22} \\ T_{33} \\ T_{23} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} T_{11} \\ T_{12} \\ T_{13} \\ T_{22} \\ T_{33} \\ T_{23} \end{pmatrix}.$$

CHAPTER 3

Closed 3-manifolds with positive Ricci curvature

In this chapter we discuss some of the ingredients that appear in the proof of Hamilton's 1982 classification of closed 3-manifolds with positive Ricci curvature. We begin by considering the maximum principle for 2-tensors, which we shall apply to the Ricci tensor to get pointwise estimates for its pinching. We then give a sketch of the derivation of the formula for the evolution of the Riemann curvature tensor under Ricci flow. In dimension 3, the associated ODE is particularly simple. The maximum principle for systems enables us to estimate the curvatures by a suitable analysis of this ODE system, which we carry out in §4. Then we discuss the gradient of scalar curvature estimate, which unlike the pointwise pinching estimates for curvature, allows us to compare curvatures at different points. Finally we state the exponential convergence results for closed 3-manifolds with positive Ricci curvature and sketch their proof.

1. The maximum principle for tensors

Since the Ricci and Riemann curvature tensors satisfy heat-type equations, just as the scalar curvature does, one can apply the maximum principle to derive estimates. Given that a symmetric 2-tensor satisfies a heat-type equation, we would like to know when the nonnegativity of the 2-tensor is preserved as time evolves. A result in this direction is provided by Hamilton's **maximum principle for tensors**. A word about notation: if α is a symmetric 2-tensor, then $\alpha \geq 0$ means that α is nonnegative definite.

THEOREM 3.1 (Maximum principle for symmetric 2-tensors). *Let $g(t)$ be a smooth 1-parameter family of Riemannian metrics on a closed manifold M^n. Let $\alpha(t)$ be a symmetric 2-tensor satisfying*

$$\frac{\partial}{\partial t}\alpha \geq \Delta_{g(t)}\alpha + \beta,$$

*where $\beta(x,t) = \beta(\alpha(x,t), g(x,t))$ is a symmetric $(2,0)$-tensor which is locally Lipschitz in all its arguments and satisfies the **null eigenvector assumption** that if A_{ij} is a nonnegative symmetric 2-tensor at a point (x,t) and if V is such that $A_{ij}V^j = 0$, then*

$$\beta_{ij}(A,g)\,V^iV^j \geq 0.$$

If $\alpha(0) \geq 0$, then $\alpha(t) \geq 0$ for all $t \geq 0$ as long as the solution exists.

PROOF. (*Idea.*) Suppose that (x_1, t_1) is a point where there exists a vector V such that $\left(\alpha_{ij} V^j\right)(x_1, t_1) = 0$ for the first time (so that $\left(\alpha_{ij} W^i W^j\right)(x, t) \geq 0$ for all W, $x \in M$, and $t \leq t_1$.) Choose V to be constant in time. We then have at (x_1, t_1)

$$\frac{\partial}{\partial t}\left(\alpha_{ij} V^i V^j\right) = \left(\frac{\partial}{\partial t}\alpha_{ij}\right) V^i V^j \geq \left(\Delta \alpha_{ij}\right) V^i V^j + \beta_{ij} V^i V^j$$
$$\geq \left(\Delta \alpha_{ij}\right) V^i V^j.$$

To handle the last term we extend V in a neighborhood of x_1 by parallel translating it along geodesics (with respect to the metric $g(t_1)$) emanating from x_1. It is easy to see that $\nabla V(x_1, t_1) = 0$ and it can also be shown that $\Delta V(x_1, t_1) = 0$. Thus we have

$$\frac{\partial}{\partial t}\left(\alpha_{ij} V^i V^j\right) \geq \left(\Delta \alpha_{ij}\right) V^i V^j = \Delta\left(\alpha_{ij} V^i V^j\right) \geq 0.$$

This shows that when α attains a zero eigenvalue for the first time, it wants to increase in the direction of any corresponding zero eigenvector. Although this does not quite complete the proof, we can make the argument rigorous by adding in an $\varepsilon > 0$ just like for the scalar maximum principle. □

EXERCISE 3.2. *Give a rigorous proof of Theorem 3.1. First show that there exists $\delta > 0$ such that $\alpha \geq 0$ on $[0, \delta]$ by applying the above argument to the symmetric 2-tensor*

$$A_\varepsilon(t) \doteqdot \alpha(t) + \varepsilon(\delta + t)\, g(t)$$

for $\varepsilon > 0$ sufficiently small and then letting $\varepsilon \to 0$.

So in order to prove that the **nonnegativity of the Ricci tensor is preserved** under the Ricci flow, all we need to do is to show that at any point and time where $R_{ij} W^i W^j \geq 0$ for all W and $R_{ij} V^j = 0$ for some V, we have

$$\left(R_{kij\ell} R_{k\ell} - R_{ik} R_{jk}\right) V^i V^j \geq 0.$$

Unfortunately, when $n \geq 4$ this is not possible in general. The main reason for this is that the Riemann curvature tensor cannot be recovered solely from the Ricci tensor (indeed, this is why the Weyl tensor does not vanish in general when $n \geq 4$). The exception to this is when $n = 3$, in which case we have

$$R_{ijk\ell} = R_{i\ell} g_{jk} + R_{jk} g_{i\ell} - R_{ik} g_{j\ell} - R_{j\ell} g_{ik} - \frac{R}{2}\left(g_{i\ell} g_{jk} - g_{ik} g_{j\ell}\right).$$

Substituting this into (2.37), we obtain the following.

LEMMA 3.3 (3d evolution of Ricci). *If $n = 3$, then under the Ricci flow we have*

(3.1)

$$\boxed{\frac{\partial}{\partial t} R_{ij} = \Delta R_{ij} + 3 R R_{ij} - 6 R_{ip} R_{jp} + \left(2\left|\mathrm{Rc}\right|^2 - R^2\right) g_{ij}. \quad (n = 3)}$$

This is a **reaction-diffusion equation**. In particular, the Laplacian term is the diffusion term whereas the rest forms the reaction term. Note that the reaction term on the RHS is essentially a quadratic in the Ricci tensor (we are also using the metric to perform contractions). By the maximum principle for tensors we have:

COROLLARY 3.4 (Hamilton 1982 - Nonnegative Ricci is preserved). *If $n = 3$ and if $\left(M^3, g(t)\right)$ is a solution to the Ricci flow on a closed manifold with* $\mathrm{Rc}(g(0)) \geq 0$, *then* $\mathrm{Rc}(g(t)) \geq 0$ *for all $t \geq 0$ as long as the solution exists.*

PROOF. We easily check that the tensor

$$\beta_{ij} = 3RR_{ij} - 6R_{ip}R_{jp} + \left(2\left|\mathrm{Rc}\right|^2 - R^2\right)g_{ij}$$

satisfies the null eigenvector assumption with respect to $\alpha_{ij} = R_{ij}$. In particular, if at a point and time Rc has a null-eigenvector V (we do not need $\mathrm{Rc} \geq 0$ here), then $2\left|\mathrm{Rc}\right|^2 - R^2 \geq 0$ and

$$\beta_{ij}V^iV^j = \left(2\left|\mathrm{Rc}\right|^2 - R^2\right)\left|V\right|^2 \geq 0.$$

□

EXERCISE 3.5 (Preservation of Ricci pinching).

(1) *Show that nonnegative sectional curvature $\frac{1}{2}Rg_{ij} - R_{ij} \geq 0$ is preserved under the Ricci flow on a closed 3-manifold.*
(2) *Show that if $R > 0$, then the inequality $R_{ij} \geq \varepsilon Rg_{ij}$ is preserved for any $\varepsilon \geq 0$ (of course, $\varepsilon \leq 1/3$.)*

2. Hamilton's 1982 theorem

Now we return to the theorem which started Ricci flow. First note that by (2.11) the **evolution of the volume form** is given by

$$\frac{\partial}{\partial t}d\mu = -Rd\mu \tag{3.2}$$

and $\mathrm{Vol}(g) \doteqdot \int_{M^n} d\mu$ evolves by

$$\frac{d}{dt}\mathrm{Vol}(g(t)) = -\int_M Rd\mu. \tag{3.3}$$

Since this is not zero in general, we modify (normalize) the Ricci flow equation to make the volume constant. In particular, we define the **normalized Ricci flow** by

$$\frac{\partial}{\partial t}\hat{g}_{ij} = -2\hat{R}_{ij} + \frac{2}{n}\hat{r}\hat{g}_{ij} \tag{3.4}$$

where $\hat{r} = \mathrm{Vol}(\hat{g})^{-1} \cdot \int_{M^n} \hat{R}\,d\hat{\mu}$ is the **average scalar curvature**. We then have (again use (2.11))

$$\frac{d}{dt}\mathrm{Vol}(\hat{g}(t)) = 0 \tag{3.5}$$

under the normalized Ricci flow. Given a solution $g(t)$, $t \in [0,T)$, of the Ricci flow, the metrics $\hat{g}(\hat{t}) \doteqdot c(t)\, g(t)$, where

$$c(t) \doteqdot \exp\left(\frac{2}{n}\int_0^t r(\tau)\, d\tau\right), \qquad \hat{t}(t) \doteqdot \int_0^t c(\tau)\, d\tau$$

are a solution of the normalized Ricci flow with $\hat{g}(0) = g(0)$. Hence solutions of the normalized Ricci flow differ from solutions of the Ricci flow only by rescalings in space and time.

THEOREM 3.6 (Hamilton 1982 - 3-manifolds with positive Ricci curvature). *Let (M^3, g_0) be a closed Riemannian 3-manifold with positive Ricci curvature. Then there exists a unique solution $g(t)$ of the normalized Ricci flow with $g(0) = g_0$ for all $t \geq 0$. Furthermore, as $t \to \infty$, the metrics $g(t)$ converge exponentially fast in every C^m-norm to a C^∞ metric g_∞ with constant positive sectional curvature.*

3. Evolution of curvature

The idea of the proof is to get estimates for various geometric quantities associated to the evolving metric, such as the curvature and its derivatives, which will show the metric limits to a constant positive sectional curvature metric. We now describe some of these estimates. In order to do so, we take a more general view point.

LEMMA 3.7 (Evolution of Rm). *The **evolution of the Riemann curvature tensor** is given by:*

$$\frac{\partial}{\partial t} R_{ijk\ell} = \Delta R_{ijk\ell} + 2\left(B_{ijk\ell} - B_{ij\ell k} + B_{ikj\ell} - B_{i\ell jk}\right) - \left(R_{ip}R_{pjk\ell} + R_{jp}R_{ipk\ell} + R_{kp}R_{ijp\ell} + R_{\ell p}R_{ijkp}\right), \tag{3.6}$$

where

$$B_{ijk\ell} \doteqdot -g^{pr}g^{qs}R_{ipjq}R_{kr\ell s} = -R_{pijq}R_{q\ell kp}. \tag{3.7}$$

The reason that there are four like terms for both the B and $\mathrm{Rc} * \mathrm{Rm}$ terms is that Rm satisfies the basic symmetry properties $R_{ijk\ell} = -R_{jik\ell} = R_{k\ell ij}$, etc. Although the formula looks complicated, the derivation is actually straightforward. We briefly sketch how it goes.

PROOF. *(Sketch.)* From (2.22) we see that

$$\boxed{\frac{\partial}{\partial t} R_{ijk}^{\ell} = \nabla_i\left(\frac{\partial}{\partial t}\Gamma_{jk}^{\ell}\right) - \nabla_j\left(\frac{\partial}{\partial t}\Gamma_{ik}^{\ell}\right).} \tag{3.8}$$

Then substituting (2.16) into this yields

$$\begin{aligned}\frac{\partial}{\partial t} R_{ijk\ell} &= \frac{\partial}{\partial t} R_{ijk}^{\ell} + \left(\frac{\partial}{\partial t} g_{\ell p}\right) R_{ijk}^{p} \\ &= -\nabla_i\nabla_j R_{k\ell} - \nabla_i\nabla_k R_{j\ell} + \nabla_i\nabla_\ell R_{jk} + \nabla_j\nabla_i R_{k\ell} \\ &\quad + \nabla_j\nabla_k R_{i\ell} - \nabla_j\nabla_\ell R_{ik} - 2R_{\ell p}R_{ijkp}.\end{aligned} \tag{3.9}$$

This doesn't look anything like a heat-type equation. Fortunately the second Bianchi identity enables us to convert this into the desired heat-type equation (3.6). In particular, we start out by

$$\Delta R_{ijk\ell} = \nabla_p \nabla_p R_{ijk\ell} = -\nabla_p \nabla_i R_{jpk\ell} - \nabla_p \nabla_j R_{pik\ell}.$$

Next we commute the p index closer to Rm

$$\Delta R_{ijk\ell} = -\nabla_i \nabla_p R_{jpk\ell} - \nabla_j \nabla_p R_{pik\ell} + \mathrm{Rm} * \mathrm{Rm}, \tag{3.10}$$

where $\mathrm{Rm} * \mathrm{Rm}$ denotes a 4-tensor quadratic in Rm. Next we apply the second Bianchi identity again to get

$$\begin{aligned} \Delta R_{ijk\ell} &= \nabla_i \nabla_k R_{jp\ell p} + \nabla_i \nabla_\ell R_{jppk} + \nabla_j \nabla_k R_{pi\ell p} \\ &\quad + \nabla_j \nabla_\ell R_{pipk} + \mathrm{Rm} * \mathrm{Rm} \qquad (3.11) \\ &= -\nabla_i \nabla_k R_{j\ell} + \nabla_i \nabla_\ell R_{jk} + \nabla_j \nabla_k R_{i\ell} \\ &\quad - \nabla_j \nabla_\ell R_{ik} + \mathrm{Rm} * \mathrm{Rm} \qquad (3.12) \end{aligned}$$

Comparing this with (3.9) yields a formula of the form

$$\frac{\partial}{\partial t} R_{ijk\ell} = \Delta R_{ijk\ell} + \mathrm{Rm} * \mathrm{Rm} + \mathrm{Rc} * \mathrm{Rm}.$$

A detailed version of this calculation yields (3.6). See for example [**153**], §6.1.3, p. 177ff. We also compute a more general version of equation (3.6) in detail in Chapter 9 when we consider the evolution of the space-time Riemann curvature tensor. □

Exercise 3.8. *Give a complete proof of (3.6). In particular, show that* $\mathrm{Rm} * \mathrm{Rm}$ *in (3.10), (3.11) and (3.12) are given by*

$$\begin{aligned} \mathrm{Rm} * \mathrm{Rm} = &R_{pijq} R_{qpk\ell} - R_{iq} R_{jqk\ell} + R_{pikq} R_{jpq\ell} + R_{pi\ell q} R_{jpkq} \\ &- R_{pjiq} R_{qpk\ell} + R_{jq} R_{iqk\ell} - R_{pjkq} R_{ipq\ell} - R_{pj\ell q} R_{ipkq}. \end{aligned}$$

Then use the definition of $B_{ijk\ell}$ *and the first Bianchi identities in a suitable way together with the fact that two terms in (3.9):*

$$-\nabla_i \nabla_j R_{k\ell} + \nabla_j \nabla_i R_{k\ell}$$

yield a quadratic curvature term.

Exercise 3.9 (Variation of Rm). *Show that if* $\frac{\partial}{\partial s} g_{ij} = v_{ij}$, *then*

$$\frac{\partial}{\partial s} R_{ijk}^{\ell} = \frac{1}{2} g^{\ell p} \left\{ \begin{array}{c} \nabla_i \nabla_j v_{kp} + \nabla_i \nabla_k v_{jp} - \nabla_i \nabla_p v_{jk} \\ -\nabla_j \nabla_i v_{kp} - \nabla_j \nabla_k v_{ip} + \nabla_j \nabla_p v_{ik} \end{array} \right\} \tag{3.13}$$

$$\begin{aligned} &= \frac{1}{2} g^{\ell p} \left(\nabla_i \nabla_k v_{jp} - \nabla_i \nabla_p v_{jk} - \nabla_j \nabla_k v_{ip} + \nabla_j \nabla_p v_{ik} \right) \qquad (3.14) \\ &\quad - \frac{1}{2} g^{\ell p} \left(R_{ijkq} v_{qp} + R_{ijpq} v_{kq} \right). \end{aligned}$$

The Riemann curvature tensor may be considered as an operator

$$\mathrm{Rm} : \wedge^2 M^n \to \wedge^2 M^n$$

defined by

$$\mathrm{Rm}\,(\alpha)_{ij} \doteqdot R_{ijk\ell}\alpha_{\ell k}. \tag{3.15}$$

DEFINITION 3.10. *We call* Rm *the* ***Riemann curvature operator*** *(or simply* ***curvature operator****). We say that* (M^n, g) *has* ***positive (nonnegative) curvature operator*** *if the eigenvalues of* Rm *are positive (nonnegative), and we denote this by* $\mathrm{Rm} > 0$ *(*$\mathrm{Rm} \geq 0$*).*

We can define the square of Rm by $\mathrm{Rm}^2 = \mathrm{Rm} \circ \mathrm{Rm} : \wedge^2 M^n \to \wedge^2 M^n$. It is interesting that there is another quadratic, like a square, which is relevant to the evolution of Rm. To describe this, we recall the Lie algebra structure on $\wedge^2 M^n$ defined by $[U,V]_{ij} \doteqdot g^{k\ell}(U_{ik}V_{\ell j} - V_{ik}U_{\ell j})$ for $U, V \in \wedge^2 M^n$. Then $\wedge^2 M^n \cong \mathfrak{so}(n)$. Choose a basis $\{\varphi^\alpha\}$ of $\wedge^2 M^n$ and let $C_\gamma^{\alpha\beta}$ denote the structure constants defined by $\left[\varphi^\alpha, \varphi^\beta\right] \doteqdot \sum_\gamma C_\gamma^{\alpha\beta}\varphi^\gamma$. We define the **Lie algebra square** $\mathrm{Rm}^\# : \wedge^2 M^n \to \wedge^2 M^n$ by

$$(\mathrm{Rm}^\#)_{\alpha\beta} \doteqdot C_\alpha^{\gamma\delta} C_\beta^{\varepsilon\zeta}\,\mathrm{Rm}_{\gamma\varepsilon}\,\mathrm{Rm}_{\delta\zeta}.$$

Note that if we choose $\{\varphi^\alpha\}$ so that Rm is diagonal, then for any 2-form η, we have $(\mathrm{Rm}^\#)_{\alpha\beta}\eta^\alpha\eta^\beta = \left(C_\alpha^{\gamma\delta}\eta^\alpha\right)^2 \mathrm{Rm}_{\gamma\gamma}\,\mathrm{Rm}_{\delta\delta}$. Hence, we see that if $\mathrm{Rm} \geq 0$, then $\mathrm{Rm}^\# \geq 0$.

We have the following nice form for the evolution equation for Rm.

LEMMA 3.11 (Evolution of the curvature operator).

$$\boxed{\frac{\partial}{\partial t}\mathrm{Rm} = \Delta\,\mathrm{Rm} + \mathrm{Rm}^2 + \mathrm{Rm}^\#.} \tag{3.16}$$

Actually we have cheated a little bit; the actual equations include additional terms of the form $\mathrm{Rc} * \mathrm{Rm}$. By using what is known as **Uhlenbeck's trick**, one obtains (3.16). The idea is to choose a vector bundle $E \to M$ isomorphic to the tangent bundle $TM \to M$ and a bundle isomorphism $\iota_0 : E \to TM$. Pulling back the initial metric we get a bundle metric $h \doteqdot \iota_0^*(g_0)$ on E. By using the metric g to identify TM and T^*M, we may consider the Ricci tensor as a bundle map $\mathrm{Rc} : TM \to TM$. We define a one parameter family of bundle isomorphisms $\iota(t) : E \to TM$ by the ODE

$$\begin{aligned} \frac{d}{dt}\iota &= \mathrm{Rc} \circ \iota \\ \iota(0) &= \iota_0. \end{aligned} \tag{3.17}$$

An easy computation shows that $\frac{\partial}{\partial t}\left[\iota(t)^* g(t)\right] = 0$. Hence $h = \iota(t)^* g(t)$ is independent of t. Using the bundle isomorphisms $\iota(t)$ we can pull back tensors on M. In particular, we consider $\iota(t)^* \mathrm{Rm}\,[g(t)]$, which is a section of $\wedge^2 E^* \otimes_S \wedge^2 E^*$. It is this tensor which satisfies (3.16), which is equivalent to:

$$\left(\frac{\partial}{\partial t} - \Delta\right) R_{abcd} = 2(B_{abcd} - B_{abdc} + B_{acbd} - B_{adbc}) \tag{3.18}$$

where R_{abcd} are the components of $\iota(t)^* \operatorname{Rm}[g(t)]$, and

$$B_{abcd} \doteqdot -R_{aebf} R_{cedf}. \tag{3.19}$$

See [**257**] or [**153**], p. 180ff for the proof.

Now we get to what is nice about dimension 3. Here, the Lie algebra structure of $\mathfrak{so}(3) \cong \mathbb{R}^3$ is very simple, namely $[U, V] = U \times V$ is the cross product. This implies $\operatorname{Rm}^\#$ is the adjoint of Rm. If we diagonalize:

$$\operatorname{Rm} = \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \mu & 0 \\ 0 & 0 & \nu \end{pmatrix},$$

then $\operatorname{Rm}^2 + \operatorname{Rm}^\#$ is also diagonal and

$$\operatorname{Rm}^2 + \operatorname{Rm}^\# = \begin{pmatrix} \lambda^2 + \mu\nu & 0 & 0 \\ 0 & \mu^2 + \lambda\nu & 0 \\ 0 & 0 & \nu^2 + \lambda\mu \end{pmatrix}. \tag{3.20}$$

So we understand the evolution equation for Rm in dimension 3 pretty well.

EXERCISE 3.12. *Derive (3.20) from (3.16).*

EXERCISE 3.13 ($n = 3$ - principal sectional curvatures). *When* $\dim M = 3$, *show that at each point there exists an orthonormal frame* $\{e_1, e_2, e_3\}$ *such that the 2-forms* $\varphi_1 \doteqdot e_2^* \wedge e_3^*$, $\varphi_2 \doteqdot e_3^* \wedge e_1^*$, $\varphi_3 \doteqdot e_1^* \wedge e_2^*$ *are eigenvectors of* Rm. *In this case* $\lambda = 2\langle \operatorname{Rm}(e_2, e_3) e_3, e_2\rangle$, $\mu = 2\langle \operatorname{Rm}(e_1, e_3) e_3, e_1\rangle$, $\nu = 2\langle \operatorname{Rm}(e_1, e_2) e_2, e_1\rangle$ *are twice the sectional curvatures.*

EXERCISE 3.14. *Show that if g has constant sectional curvature, then* $\operatorname{Rm} \equiv \frac{2R}{n(n-1)} \operatorname{Id}_{\wedge^2}$.

EXERCISE 3.15. *By examining the evolution of the Einstein tensor* $\frac{1}{2}Rg_{ij} - R_{ij}$ *(whose eigenvalues are the principal sectional curvatures), verify (3.16) where* $\operatorname{Rm}^2 + \operatorname{Rm}^\#$ *is given by (3.20) when* $n = 3$. *(Compare with Exercise 3.5.) Note that one needs to apply Uhlenbeck's trick to the evolution equation for* $\frac{1}{2}Rg - \operatorname{Rc}$ *to get the exact correspondence.*

In dimension 4 we can orthogonally decompose $\Lambda^2 = \Lambda^2_+ \oplus \Lambda^2_-$ into its **self-dual** and **anti-self-dual** subspaces, which are the eigenspaces of the Hodge star operator with eigenvalues 1 and -1, respectively. This gives a block decomposition of Rm as

$$\operatorname{Rm} = \begin{pmatrix} A & B \\ {}^tB & C \end{pmatrix},$$

where $A : \Lambda^2_+ \to \Lambda^2_+$, $C : \Lambda^2_- \to \Lambda^2_-$ and $B : \Lambda^2_- \to \Lambda^2_+$ (essentially A, B, C are 3×3 matrices) and A, C are self-adjoint (symmetric). We can compute

$$\mathrm{Rm}^{\#} = 2 \begin{pmatrix} A^{\#} & B^{\#} \\ (B^t)^{\#} & C^{\#} \end{pmatrix},$$

where $A^{\#}, B^{\#}, C^{\#}$ are the adjoints of the 3×3 matrices, i.e.,

$$\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & k \end{pmatrix}^{\#} = \begin{pmatrix} ek - fh & fg - dk & dh - eg \\ ch - bk & ak - cg & bg - ah \\ bf - ce & cd - af & ae - bd \end{pmatrix}.$$

This can also be seen from the splitting of the Lie algebra $\mathfrak{so}(4)$ as the direct sum of two copies of $\mathfrak{so}(3)$, which is why dimension four can is more tractable than dimensions 5 and higher. The evolution of the curvature breaks up into the three systems of equations:

$$\begin{aligned} \frac{\partial}{\partial t} A &= \Delta A + A^2 + 2A^{\#} + BB^t \\ \frac{\partial}{\partial t} B &= \Delta B + AB + BC + 2B^{\#} \\ \frac{\partial}{\partial t} C &= \Delta C + C^2 + 2C^{\#} + B^t B, \end{aligned} \tag{3.21}$$

where $B^t : \Lambda^2_+ \to \Lambda^2_-$ is the transpose (i.e., $\langle Bx, y \rangle = \langle x, B^t y \rangle$.) Note that the Bianchi identity implies the following equality of traces: $\operatorname{tr} A = \operatorname{tr} C$.

Primarily by performing a suitable analysis of the system of ODE (3.21) Hamilton used Ricci flow to classify closed 4-manifolds with positive curvature operator [**257**].

THEOREM 3.16 (Hamilton 1986 - 4-manifolds with positive curvature operator). *If (M^n, g_0) is a closed 4-manifold with positive curvature operator, then there exists a smooth solution $g(t)$ to the normalized Ricci flow with $g(0) = g_0$ and defined for all $t \in [0, \infty)$. As $t \to \infty$, the solution converges exponentially in every C^k norm to a constant positive sectional metric. In particular, M^n is diffeomorphic to either S^4 or $\mathbb{RP}^4$.*

4. The maximum principle for systems

Now that we know that the nonnegativity of the Ricci tensor is preserved under the Ricci flow on closed 3-manifolds, we are interested in a more precise understanding of the Ricci tensor as the metric evolves. A very useful tool is the maximum principle for tensors as discussed in the previous section. This principle has been abstracted to the following setting.

Recall that Rm is a section of the bundle $\pi : E \to M$, where $E \doteqdot \wedge^2 M^n \otimes_S \wedge^2 M^n$. This bundle has a natural bundle metric and connection induced by the Riemannian metric and connection on TM. Let $E_x \doteqdot \pi^{-1}(x)$ be the fiber over x. For each $x \in M$, consider the system of ODE on E_x

corresponding to the PDE (3.16) obtained by dropping the Laplacian term:

$$\boxed{\frac{d}{dt}\mathbf{M} = \mathbf{M}^2 + \mathbf{M}^{\#}} \tag{3.22}$$

where $\mathbf{M} \in E_x$ is a symmetric $N \times N$ matrix, where $N = \frac{n(n-1)}{2} = \dim \mathfrak{so}(n)$. The **maximum principle for systems** (for the proof of a more general version which applies to sections of vector bundles satisfying heat-type equations, see section 4 of [**257**] or [**143**]) says the following. A set K in a vector space is said to be **convex** if for any $X, Y \in K$, we have $sX + (1-s)Y \in K$ for all $s \in [0,1]$. A subset K of the vector bundle E is said to be **invariant under parallel translation** if for every path $\gamma : [a,b] \to M$ and vector $X \in K \cap E_{\gamma(a)}$, the unique parallel section $X(s) \in E_{\gamma(s)}$, $s \in [a,b]$, along $\gamma(s)$ with $X(a) = X$ is contained in K.

THEOREM 3.17 (Maximum principle systems applied to the curvature operator). *Let $g(t)$, $t \in [0,T)$, be a solution to the Ricci flow on a closed manifold M^n. Let $K \subset E$ be a subset which is invariant under parallel translation and whose intersection $K_x \doteqdot K \cap E_x$ with each fiber is closed and convex. Suppose the* ODE *(3.22) has the property that for any $\mathbf{M}(0) \in K$, we have $\mathbf{M}(t) \in K$ for all $t \in [0,T)$. If $\operatorname{Rm}(0) \in K$, then $\operatorname{Rm}(t) \in K$ for all $t \in [0,T)$.*

Since if $\operatorname{Rm} \geq 0$, then $\operatorname{Rm}^2 \geq 0$ and $\operatorname{Rm}^{\#} \geq 0$, by (3.16) and the above theorem, we have the following.

COROLLARY 3.18 ($\operatorname{Rm} \geq 0$ is preserved). *If $(M^n, g(t))$, $t \in [0,T)$, is a solution to the Ricci flow on a closed manifold with $\operatorname{Rm}(g(0)) \geq 0$, then $\operatorname{Rm}(g(t)) \geq 0$ for all $t \in [0,T)$.*

In dimension 3, if $\mathbf{M}(0)$ is diagonal, then $\mathbf{M}(t)$ remains diagonal. Let $\lambda_1(\mathbf{M}) \leq \lambda_2(\mathbf{M}) \leq \lambda_3(\mathbf{M})$ be the eigenvalues of $\mathbf{M}$. Under the ODE the ordering of the eigenvalues is preserved and we have

$$\begin{aligned} \frac{d\lambda_1}{dt} &= \lambda_1^2 + \lambda_2\lambda_3 \\ \frac{d\lambda_2}{dt} &= \lambda_2^2 + \lambda_1\lambda_3 \\ \frac{d\lambda_3}{dt} &= \lambda_3^2 + \lambda_1\lambda_2. \end{aligned} \tag{3.23}$$

With this setup, we can come up with a number of closed, fiberwise convex sets K, invariant under parallel translation, which are preserved by the ODE. Each such set corresponds to an a priori estimate for the curvature Rm.

The following sets $K \subset E$ are invariant under parallel translation and for each $x \in M$, K_x is closed, convex and preserved by the ODE (3.23).

(1) Given $C_0 \in \mathbb{R}$, let $K = \{\mathbf{M} : \lambda_1(\mathbf{M}) + \lambda_2(\mathbf{M}) + \lambda_3(\mathbf{M}) \geq C_0\}$. The trace $\lambda_1 + \lambda_2 + \lambda_3 : E_x \to \mathbb{R}$ is a linear function, which implies

that K is closed and convex. That K is preserved by the ODE (3.23) follows from

$$\frac{d}{dt}(\lambda_1+\lambda_2+\lambda_3) = \frac{1}{2}\left[(\lambda_1+\lambda_2)^2+(\lambda_1+\lambda_3)^2+(\lambda_2+\lambda_3)^2\right]$$

$$\geq \frac{2}{3}(\lambda_1+\lambda_2+\lambda_3)^2 \geq 0. \tag{3.24}$$

(We interjected the first inequality since we will find it useful later.) Hence:

(**Lower bound of scalar is preserved**) if $R \geq C_0$ at $t=0$ for some $C_0 \in \mathbb{R}$, then

$$\boxed{R \geq C_0} \tag{3.25}$$

for all $t \geq 0$. This is something we have already seen in Corollary 2.10.

(2) Let $K = \{\mathbf{M} : \lambda_1(\mathbf{M}) \geq 0\}$. Each K_x is closed and convex since $\lambda_1 : E_x \to \mathbb{R}$ is a concave function. Indeed, $\lambda_1(\mathbf{M}) = \min_{|V|=1} \mathbf{M}(V,V)$ so that

$$\lambda_1(s\mathbf{M}_1 + (1-s)\mathbf{M}_2) \geq s\lambda_1(\mathbf{M}_1) + (1-s)\lambda_1(\mathbf{M}_2)$$

for all $s \in [0,1]$. We see that K is preserved by the ODE since

$$\frac{d\lambda_1}{dt} = \lambda_1^2 + \lambda_2\lambda_3 \geq 0$$

whenever $\lambda_1 \geq 0$. That is, if $\mathbf{M}(t)$ is a solution of the ODE (3.23) with $\lambda_1(\mathbf{M}(0)) \geq 0$, then $\lambda_1(\mathbf{M}(t)) \geq 0$ for all $t \geq 0$. This implies (this is a special case of Corollary 3.18):

(**Nonnegative sectional curvature is preserved**) The condition

$$\boxed{\mathrm{Rm} \geq 0} \tag{3.26}$$

is preserved under the Ricci flow. Since any 2-form on a 3-manifold is the wedge product of two 1-forms, this is equivalent to the sectional curvature being nonnegative.

(3) Let $K = \{\mathbf{M} : \lambda_1(\mathbf{M}) + \lambda_2(\mathbf{M}) \geq 0\}$. Since $\lambda_1 + \lambda_2$ is concave:

$$(\lambda_1+\lambda_2)(\mathbf{M}) = \min\{\mathbf{M}(V_1,V_1) + \mathbf{M}(V_2,V_2) : \{V_1,V_2\} \text{ orthonormal}\},$$

we have K is closed and convex. We compute

$$\frac{d}{dt}(\lambda_1+\lambda_2) = \lambda_1^2 + \lambda_2^2 + (\lambda_1+\lambda_2)\lambda_3 \geq 0$$

whenever $\lambda_1 + \lambda_2 \geq 0$. From this we see that:

(**Nonnegative Ricci is preserved**)

$$\mathrm{Rc} \geq 0 \tag{3.27}$$

is preserved under the Ricci flow since the smallest eigenvalue of Rc is $\lambda_1(\mathrm{Rm}) + \lambda_2(\mathrm{Rm})$.

(4) Given $C \geq 1/2$, let

$$K = \{\mathbf{M} : \lambda_3(\mathbf{M}) \leq C(\lambda_1(\mathbf{M}) + \lambda_2(\mathbf{M}))\}.$$

Since λ_3 is convex ($\lambda_3(\mathbf{M}) = \max_{|V|=1} \mathbf{M}(V,V)$) and $\lambda_1 + \lambda_2$ is concave, we have K_x is convex for all $x \in M$. That each K_x is preserved by the ODE follows from the calculation:

$$\frac{d}{dt}[\lambda_3 - C(\lambda_1 + \lambda_2)] = \lambda_3(\lambda_3 - C(\lambda_1 + \lambda_2)) - C\left(\lambda_1^2 - \frac{1}{C}\lambda_1\lambda_2 + \lambda_2^2\right).$$

In particular, if $\lambda_3 - C(\lambda_1 + \lambda_2) = 0$ and $C \geq 1/2$, then

$$\frac{d}{dt}[\lambda_3 - C(\lambda_1 + \lambda_2)] \leq 0.$$

Suppose $\mathrm{Rc}(g(0)) > 0$. Since M^3 is compact, there exists $C \geq 1/2$ such that at $t = 0$

$$\lambda_3(\mathrm{Rm}) \leq C(\lambda_1(\mathrm{Rm}) + \lambda_2(\mathrm{Rm})). \tag{3.28}$$

That is, $\mathrm{Rm}(g(0)) \subset K$. By the maximum principle for tensors, $\mathrm{Rm}(g(t)) \subset K$ and inequality (3.28) is true for all $t \geq 0$. Now (3.28) implies $\mathrm{Rc} \geq C^{-1}\lambda_3(\mathrm{Rm})\, g \geq \frac{1}{3}C^{-1}Rg$. Thus:

(**Ricci pinching is preserved**) there exists a constant $\varepsilon > 0$ such that

$$\boxed{\mathrm{Rc} \geq \varepsilon R g. \quad (n = 3)} \tag{3.29}$$

In particular, when M^3 is compact, we have that $\mathrm{Rc} > 0$ is preserved. (Compare with Exercise 3.5.)

Remark. Note that if $(\lambda_1(\mathrm{Rm}) + \lambda_2(\mathrm{Rm}))(x_0, t_0) < 0$ for some $(x_0, t_0) \in M^3 \times [0, T)$, then since $C \geq 1/2$, we have $\lambda_3(\mathrm{Rm}) = \lambda_1(\mathrm{Rm}) = \lambda_2(\mathrm{Rm})$ at (x_0, t_0). Since $\lambda_1 + \lambda_2 < 0$ holds on a connected neighborhood U of x_0 at time t_0, we have $\lambda_1 = \lambda_2 = \lambda_3 = \frac{R}{3}$ in U (recall that λ_i are twice the sectional curvatures). By the contracted Bianchi identity, we then have R is constant on U. Since M^3 is connected, it is easy to conclude that $\lambda_1 = \lambda_2 = \lambda_3 = \frac{R}{3}$ on all of M, where the scalar curvature R is a negative constant. Thus, if $\mathrm{Rm}(g(t_0)) \subset K$ for some t_0 and if $g(t_0)$ does not have constant negative sectional curvature, then $\mathrm{Rc} \geq 0$.

(5) Given $C_0 > 0$, $C_1 \geq 1/2$, $C_2 < \infty$ and $\delta > 0$, let

$$K = \left\{ \mathbf{M} : \begin{array}{c} \lambda_3(\mathbf{M}) - \lambda_1(\mathbf{M}) - C_2(\lambda_1(\mathbf{M}) + \lambda_2(\mathbf{M}) + \lambda_3(\mathbf{M}))^{1-\delta} \leq 0 \\ \lambda_3(\mathbf{M}) \leq C_1(\lambda_1(\mathbf{M}) + \lambda_2(\mathbf{M})) \\ \lambda_1(\mathbf{M}) + \lambda_2(\mathbf{M}) + \lambda_3(\mathbf{M}) \geq C_0 \end{array} \right\}.$$

K is a convex set since $\lambda_3 - \lambda_1 - C_2(\lambda_1 + \lambda_2 + \lambda_3)^{1-\delta}$ is a convex function for $C_2 > 0$. Observe that if $\mathbf{M} \in K$, then $\lambda_1(\mathbf{M}) + \lambda_2(\mathbf{M}) > 0$ by the last two inequalities in the definition of K. We have already seen that the inequalities $\lambda_1 + \lambda_2 + \lambda_3 \geq C_0$ and

$\lambda_3 \leq C_1 (\lambda_1 + \lambda_2)$ are preserved under the ODE. Since $C_0 > 0$, we can compute

$$\begin{aligned}
&\frac{d}{dt} \log \left(\frac{\lambda_3 - \lambda_1}{(\lambda_1 + \lambda_2 + \lambda_3)^{1-\delta}} \right) \\
&= \delta (\lambda_1 + \lambda_3 - \lambda_2) - (1-\delta) \frac{(\lambda_1 + \lambda_2) \lambda_2 + (\lambda_2 - \lambda_1) \lambda_3 + \lambda_2^2}{\lambda_1 + \lambda_2 + \lambda_3} \\
&\leq \delta (\lambda_1 + \lambda_3 - \lambda_2) - (1-\delta) \frac{\lambda_2^2}{\lambda_1 + \lambda_2 + \lambda_3}.
\end{aligned}$$

Note that

$$\frac{\lambda_2^2}{\lambda_1 + \lambda_2 + \lambda_3} \geq \frac{1}{6} \frac{(\lambda_1 + \lambda_2) \lambda_2}{\lambda_3} \geq \frac{1}{6C_1} \lambda_2$$

since $\lambda_2 + \lambda_3 \leq 2\lambda_3 \leq 2C_1 (\lambda_1 + \lambda_2)$, and we also have

$$\lambda_1 + \lambda_3 - \lambda_2 \leq \lambda_3 \leq C_1 (\lambda_1 + \lambda_2) \leq 2C_1 \lambda_2.$$

Hence, choosing $\delta > 0$ small enough so that $\frac{\delta}{1-\delta} \leq \frac{1}{12C_1^2}$, we have

$$\frac{d}{dt} \log \left(\frac{\lambda_3 - \lambda_1}{(\lambda_1 + \lambda_2 + \lambda_3)^{1-\delta}} \right) \leq 0.$$

Since $\lambda_3 - \lambda_1 \geq \left| \mathrm{Rc} - \frac{1}{3} R g \right|$, this implies:

(**Ricci pinching improves**) there exist constants $C < \infty$ and $\delta > 0$ such that

$$\boxed{\left| \mathrm{Rc} - \frac{1}{3} R g \right| \leq C R^{1-\delta}. \quad (n = 3)} \tag{3.30}$$

EXERCISE 3.19 (3d trace-free part of Rc and Rm). *Show that when* $n = 3$,

$$\left| \mathrm{Rc} - \frac{1}{3} R g \right|^2 = \left| \mathrm{Rm} - \frac{1}{3} R \, \mathrm{Id}_{\wedge^2} \right|^2.$$

In summary, the main estimates we have proved for the curvatures are (3.25), (3.29) and (3.30).

Let $[0, T)$ denote the maximum time interval of existence of our solution. Recall that from applying the maximum principle to the evolution equation for scalar curvature $\frac{\partial R}{\partial t} \geq \Delta R + \frac{2}{3} R^2$ and our assumption that $R_{\min}(0) \geq 0$, we have

$$R_{\min}(t) \geq \frac{1}{R_{\min}(0)^{-1} - \frac{2}{3} t}.$$

Hence $T \leq \frac{3}{2} R_{\min}(0)^{-1} < \infty$. In the next section we shall prove that

$$\sup_{M \times [0,T)} |\mathrm{Rm}| = \infty. \tag{3.31}$$

Intuitively speaking, we are in good shape now. Since the Ricci curvature is positive, the metric is shrinking: $\frac{\partial}{\partial t} g = -2 \, \mathrm{Rc} < 0$. If we can show an appropriate gradient estimate for the scalar curvature, then we could conclude $\lim_{t\to T} R_{\min}(t) = \infty$. Assuming this, we then would have

$$\left| \frac{\mathrm{Rc}}{R} - \frac{1}{3} g \right| \leq C R^{-\delta},$$

which tends to 0 as $t \to T$. To the finish the proof of Theorem 3.6 we need to further show that the solution $\tilde{g}(\tilde{t})$ to the corresponding normalized Ricci flow exists for all time and the scale invariant quantity $\left| \frac{\widetilde{\mathrm{Rc}}}{\tilde{R}} - \frac{1}{3}\tilde{g} \right|$ decays exponentially to zero as $\tilde{t} \to \infty$. We refer the reader to [**255**] or [**153**], p. 194ff, for details (see the next section for the statements of the higher derivative estimates which are useful for general solutions to the Ricci flow).

REMARK 3.20 ($S^2 \times S^1$ example). *It is instructive to keep in mind the example of the round product $S^2 \times S^1$ which has nonnegative Ricci curvature but not positive Ricci curvature. Under the Ricci flow the metric remains a round product. If the initial S^2 has radius r_0, then the radius at time t is $r(t) = \sqrt{r_0^2 - t}$. The radius of the circle S^1 remains constant since the Ricci curvature in the circle direction is zero. Note that at any point and time the curvature operator takes the form*

$$\mathrm{Rm} = \begin{pmatrix} R & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

In particular,

$$\left| \mathrm{Rm} - \frac{1}{3} R \, \mathrm{Id}_{\wedge^2} \right|^2 = \frac{2}{3} R^2.$$

5. Gradient of scalar curvature estimate

A fundamental estimate used in combination with the 'Ricci pinching improves' estimate is the **gradient estimate for the scalar curvature**. By the 'Ricci pinching improves' estimate, we can apply the following result to solutions of the Ricci flow on closed 3-manifolds with positive Ricci curvature.

PROPOSITION 3.21 (∇R estimate). *Let $(M^n, g(t))$, $n \geq 3$, be a solution to the Ricci flow on a closed n-manifold with positive scalar curvature. Suppose that the solution satisfies the estimate:*

$$\left| \mathrm{Rm} - \frac{2R}{n(n-1)} \mathrm{Id}_{\wedge^2} \right| \leq K R^{1-\varepsilon} \tag{3.32}$$

for some constants $K < \infty$ and $\varepsilon > 0$. Then for every $\eta > 0$ and $\theta > 0$, there exists a constant $C = C(g_0, \eta, \theta) < \infty$ such that at any point and time $(\bar{x}, \bar{t})$ where the scalar curvature

(1) (is large enough) $R(\bar{x}, \bar{t}) \geq C$, *and*

(2) (comparable to its prior maximum) $R(\bar{x},\bar{t}) \geq \eta \max_{M^3\times[0,\bar{t}]} R$,
we have

$$|\nabla \mathrm{Rm}|(\bar{x},\bar{t}) \leq \theta R^{3/2}(\bar{x},\bar{t}).$$

REMARK 3.22.

(1) *As is usual for a priori estimates, it is important to note the scaling properties of the quantities considered. In particular, scale-invariant estimates are easier to prove. For example, the 'Ricci pinching is preserved' estimate is scale-invariant whereas the 'Ricci pinching improves' estimate is not scale-invariant. Here,* $|\nabla \mathrm{Rm}|$ *scales like* $g^{-3/2}$ *and* R *scales like* g^{-1}, *so that* $|\nabla \mathrm{Rm}|$ *and* $R^{3/2}$ *scale the same.*
(2) *The original proof in* [**255**] *(see also section 4 of* [**287**]*) is based on a maximum principle estimate. The following proof is taken from notes by M.-T. Wang of Hamilton's lectures at Harvard University during 1996-1997.*
(3) *The first proof below is a modification of Hamilton's proof which, in the absence of Perelman's no local collapsing theorem, used the exponential map to locally pull back the dilated solutions to obtain a local constant curvature limit.*

PROOF. By (3.32) and the positive scalar curvature assumption, $|\mathrm{Rm}| \leq CR$ for some constant $C < \infty$. Now suppose the proposition is false. Then there exist $\eta > 0$ and $\theta > 0$ such that for any sequence $C_i \to \infty$, there exist points and times (x_i, t_i) with

$$R(x_i,t_i) \geq \max\left\{C_i, \eta \max_{M^3\times[0,t_i]} R\right\} \tag{3.33}$$

and

$$|\nabla \mathrm{Rm}|(x_i,t_i) \geq \theta R^{3/2}(x_i,t_i). \tag{3.34}$$

By (3.33), Perelman's no local collapsing theorem, and the compactness theorem (see Theorem 5.19), there exists a subsequence such that the dilated solutions $g_i(t) = R(x_i,t_i)\, g\left(t_i + R(x_i,t_i)^{-1} t\right)$ converge in C^∞ on compact sets to a complete ancient solution $(M_\infty^n, g_\infty(t))$ to the Ricci flow with bounded curvature. $g_\infty(t)$ has positive scalar curvature and is defined on an interval $(-\infty,\omega)$, where $\omega > 0$. By (3.32), we have

$$\mathrm{Rm}(g_\infty(t)) \equiv \frac{2R(g_\infty(t))}{n(n-1)} \mathrm{Id}_{\wedge^2} \tag{3.35}$$

on $M_\infty^n \times (-\infty,\omega)$. By Exercise 1.16 (Schur's lemma), since $n \geq 3$, we have $R(g_\infty(t)) \equiv \mathrm{const}(t)$ and hence $|\nabla \mathrm{Rm}(g_\infty(t))| \equiv 0$. However (3.34) implies

$$|\nabla \mathrm{Rm}(g_\infty)|(x_\infty,0) \geq \theta R(g_\infty)^{3/2}(x_\infty,0) > 0,$$

which is a contradiction. □

We remark that the original proof in [**255**] of the estimate for the gradient of the scalar curvature used the contracted Bianchi identity in the following way. Decomposing the 3-tensor $\nabla_i R_{jk}$ into its irreducible components, we let

$$\nabla_i R_{jk} \doteqdot E_{ijk} + F_{ijk}$$

where

$$E_{ijk} \doteqdot \frac{1}{20}\left(\nabla_j R g_{ik} + \nabla_k R g_{ij}\right) + \frac{3}{10}\nabla_i R g_{jk}. \tag{3.36}$$

Then $\langle E_{ijk}, F_{ijk}\rangle = 0$, $|E_{ijk}|^2 = \frac{7}{20}|\nabla_i R|^2$ and

$$|\nabla_i R_{jk}|^2 = |E_{ijk}|^2 + |F_{ijk}|^2 \geq \frac{7}{20}|\nabla_i R|^2. \tag{3.37}$$

This estimate is better than the more elementary $|\nabla_i R_{jk}|^2 \geq \frac{1}{3}|\nabla_i R|^2$ which follows from the general estimate $|a_{ij}|^2 \geq \frac{1}{n}\left(g^{ij}a_{ij}\right)^2$ and $n = 3$ (see [**287**] for its generalization to higher dimensions).

Exercise 3.23 (Quicker proof with a worse constant). *By using the inequality*

$$\left|\nabla_i R_{jk} - \frac{1}{3}\nabla_i R g_{jk}\right|^2 \geq \frac{1}{3}\left|\operatorname{div}\left(\operatorname{Rc} - \frac{1}{3}Rg\right)\right|^2$$

and the contracted second Bianchi identity, show that

$$|\nabla_i R_{jk}|^2 \geq \frac{37}{108}|\nabla_i R|^2,$$

which is weaker than (3.37).

To get the gradient of scalar curvature estimate we first compute that (see [**255**], Lemma 11.3)

$$\begin{aligned}\left(\frac{\partial}{\partial t} - \Delta\right)\left(\frac{|\nabla R|^2}{R}\right) &= -\frac{2}{R^3}|R\nabla_i\nabla_j R - \nabla_i R\nabla_j R|^2 \\ &\quad + \frac{4}{R}\left\langle \nabla R, \nabla|\operatorname{Rc}|^2\right\rangle - 2\frac{|\operatorname{Rc}|^2}{R^2}|\nabla R|^2 \\ &\leq 16|\nabla \operatorname{Rc}|^2 - 2\frac{|\operatorname{Rc}|^2}{R^2}|\nabla R|^2 \end{aligned} \tag{3.38}$$

where we used $|\operatorname{Rc}| \leq R$ and $|\nabla R| \leq \sqrt{3}|\nabla \operatorname{Rc}| \leq 2|\nabla \operatorname{Rc}|$. Since

$$\left(\frac{\partial}{\partial t} - \Delta\right)(R^2) = -2|\nabla R|^2 + 4R|\operatorname{Rc}|^2,$$

and $|\operatorname{Rc}|^2 \geq \frac{1}{3}R^2$, this implies that for any $\varepsilon \leq 1/3$, we have

$$\left(\frac{\partial}{\partial t} - \Delta\right)\left(\frac{|\nabla R|^2}{R} - \varepsilon R^2\right) \leq 16|\nabla_i R_{jk}|^2 - \frac{4}{3}\varepsilon R^3.$$

To handle the bad (positive) $|\nabla_i R_{jk}|^2$ term on the RHS we bring in the equation (see Lemma 11.7 in [**255**])

$$\left(\frac{\partial}{\partial t} - \Delta\right)\left(|\mathrm{Rc}|^2 - \frac{1}{3}R^2\right) = -2\left(|\nabla_i R_{jk}|^2 - \frac{1}{3}|\nabla_i R|^2\right) - 2R^3 + \frac{26}{3}R|\mathrm{Rc}|^2 - 8\,\mathrm{Trace}_g\left(\mathrm{Rc}^3\right)$$

$$\leq -\frac{2}{21}|\nabla_i R_{jk}|^2 + 4R\left(|\mathrm{Rc}|^2 - \frac{1}{3}R^2\right) \tag{3.39}$$

which has a good $|\nabla_i R_{jk}|^2$ term on the RHS. Note that (3.37) was used to obtain the last inequality. Combining the above formulas yields

$$\left(\frac{\partial}{\partial t} - \Delta\right)\left(\frac{|\nabla R|^2}{R} - \varepsilon R^2 + 168\left(|\mathrm{Rc}|^2 - \frac{1}{3}R^2\right)\right) \leq C(\varepsilon)$$

where we use the fact that there exist $\delta > 0$ and $C < \infty$ such that $|\mathrm{Rc}|^2 - \frac{1}{3}R^2 \leq CR^{2-\delta}$ (3.30). Since the solution exists only for a finite time, we have $F \leq C$ and hence the following

PROPOSITION 3.24 (Gradient of scalar curvature estimate). *Let $\left(M^3, g(0)\right)$ be a closed 3-manifold with positive Ricci curvature. For any $\varepsilon > 0$, there exists $C(\varepsilon)$ depending only on ε and $g(0)$ such that*

$$|\nabla R|^2(x,t) \leq \varepsilon R(x,t)^3 + C(\varepsilon)$$

as long as the solution exists.

REMARK 3.25. *Note how the contracted second Bianchi identity enters in both proofs of the 'gradient of scalar curvature' estimate.*

EXERCISE 3.26 (∇R estimate again). *Let $\left(M^3, g(0)\right)$ be a closed 3-manifold with positive Ricci curvature. Prove the following variant of the gradient of scalar curvature estimate. There exist constants $\beta_0 > 0$ and $\delta > 0$ depending only on $g(0)$ such that for all $\beta \in [0, \beta_0]$*

$$\frac{|\nabla R|^2}{R^3} \leq \beta R^{-\delta} + CR^{-3} \tag{3.40}$$

where $C < \infty$ depends only on β and $g(0)$. HINT *(see section 6.6 of [**153**] for more details): Let*

$$V \doteqdot \frac{|\nabla R|^2}{R} + \frac{37}{2}\left(8\sqrt{3}+1\right)\left(|\mathrm{Rc}|^2 - \frac{1}{3}R^2\right)$$

and show that

$$\frac{\partial}{\partial t}V \leq \Delta V - |\nabla\,\mathrm{Rc}|^2 + \frac{7400\sqrt{3}+925}{3}R\left(|\mathrm{Rc}|^2 - \frac{1}{3}R^2\right)$$

$$\leq \Delta V - |\nabla\,\mathrm{Rc}|^2 + CR^{3-2\delta}$$

where we used (3.30) to get the last inequality. Then use the equation

$$\frac{\partial}{\partial t}R^{2-\delta} = \Delta\left(R^{2-\delta}\right) - (2-\delta)(1-\delta)R^{-\delta}|\nabla R|^2 + 2(2-\delta)R^{1-\delta}|\mathrm{Rc}|^2$$

to derive

$$\frac{\partial}{\partial t}\left(V - \beta R^{2-\delta}\right) \le \Delta\left(V - \beta R^{2-\delta}\right) + C$$

where C depends only on β and $g(0)$. Estimate (3.40) follows from this.

6. Curvature tends to constant

Lemma 3.27 (Global scalar curvature pinching). *We have*

$$\lim_{t\to T}\frac{R_{\max}(t)}{R_{\min}(t)} = 1. \tag{3.41}$$

In fact, there exist constants $C < \infty$ and $\gamma > 0$ depending only on $g(0)$ such that

$$\frac{R_{\min}(t)}{R_{\max}(t)} \ge 1 - CR_{\max}(t)^{-\gamma} \tag{3.42}$$

for all $t \in [0,T)$.

Remark 3.28. *Since $\lim_{t\to T} R_{\max}(t) = \infty$, (3.42) implies (3.41) and*

$$\lim_{t\to T} R_{\min}(t) = \infty.$$

Proof. By (3.40), $\inf_{M^3\times[0,T)} R > 0$, and $\lim_{t\to T} R_{\max}(t) = \infty$, there exist a constant $C < \infty$ and $\delta > 0$ such that

$$|\nabla R(x,t)| \le CR_{\max}(t)^{3/2-\delta}$$

for all $x \in M^3$ and $t \in [0,T)$. Given $t \in [0,T)$, there exists $x_t \in M^3$ such that $R_{\max}(t) = R(x_t,t)$. Given $\eta > 0$, to be chosen sufficiently small later, for any point $x \in B\left(x_t, \frac{1}{\eta\sqrt{R_{\max}(t)}}\right)$ we have

$$R_{\max}(t) - R(x,t) \le \frac{1}{\eta\sqrt{R_{\max}(t)}}\max_{M^3}|\nabla R(t)| \le \frac{C}{\eta}R_{\max}(t)^{1-\delta}$$

so that

$$R(x,t) \ge R_{\max}(t)\left(1 - \frac{C}{\eta}R_{\max}(t)^{-\delta}\right) \tag{3.43}$$

for all $x \in B\left(x_t, \frac{1}{\eta\sqrt{R_{\max}(t)}}\right)$. We claim that this ball is all of M^3, from which (3.42) follows. The argument goes like this. Since $\lim_{t\to T} R_{\max}(t) = \infty$, by (3.43), there exists $\tau < T$ such that for $t \in [\tau, T)$ we have

$$R(x,t) \ge R_{\max}(t)(1-\eta)$$

for all $x \in B\left(x_t, \frac{1}{\eta\sqrt{R_{\max}(t)}}\right)$. Now the pinching estimate $\operatorname{Rc} \geq \varepsilon R g$ where $\varepsilon > 0$, shows that for $\eta > 0$ sufficiently small $M^3 = B\left(x_t, \frac{1}{\eta\sqrt{R_{\max}(t)}}\right)$. □

LEMMA 3.29 (Global sectional curvature pinching). *For every $\varepsilon \in (0,1)$, there exists $\tau < T$ such that for all $t \in [\tau, T)$ the sectional curvatures of $g(t)$ are positive and*

$$\min_{x \in M^3} \lambda_1(\operatorname{Rm})(x,t) \geq (1-\varepsilon) \max_{x \in M^3} \lambda_3(\operatorname{Rm})(x,t).$$

PROOF. By (3.30), there exists $C < \infty$ and $\delta > 0$ such that

$$\frac{\lambda_1(\operatorname{Rm})}{\lambda_3(\operatorname{Rm})}(x,t) \geq 1 - C\frac{R^{1-\delta}}{\lambda_3(\operatorname{Rm})}(x,t) \geq 1 - 3CR_{\min}(t)^{-\delta}$$

for all $x \in M^3$ and $t \in [0,T)$. We leave it as an exercise to the reader to show that this implies that for any $\varepsilon > 0$, there exists $\tau < T$ such that for any $x, y \in M^3$ and $t \in [\tau, T)$ we have

$$\lambda_1(\operatorname{Rm})(x,t) \geq (1-\varepsilon)\lambda_3(\operatorname{Rm})(y,t). \tag{3.44}$$

The idea for deriving the comparison (3.44) is to make an intermediate comparison with the scalar curvature of the point in question, whether it be x or y. □

7. Exponential convergence to constant curvature of the normalized flow

Now we revert back to the normalized flow

$$\frac{\partial}{\partial t} g_{ij} = -2R_{ij} + \frac{2}{3} r g_{ij}.$$

This flow is equivalent to the original Ricci flow by rescaling time and space (the metrics). See §3 of [**255**] or §6.9 of [**153**] for details. It is useful to know how evolution equations change upon normalizing the Ricci flow. We say that a tensor quantity X depending on the metric g has **degree** k in g if $X(cg) = c^k X(g)$ for any $c > 0$.

EXERCISE 3.30 (Degrees of tensors). *Show that* Rm *has degree* 1, Rc *has degree* 0, *R has degree* -1, *$d\mu$ has degree $n/2$ (if* $\dim M = n$.*)*

The following is not hard to prove (see Lemma 17.1 of [**255**]).

LEMMA 3.31 (Going from unnormalized to normalized RF). *If an expression $X = X(g)$ formed algebraically from the metric and the Riemann curvature tensor by contractions has degree k and if under the Ricci flow*

$$\frac{\partial X}{\partial t} = \Delta X + Y, \tag{3.45}$$

then the degree of Y *is* $k-1$ *and the evolution under the normalized Ricci flow* $\frac{\partial}{\partial t}\tilde{g}_{ij} = -2\tilde{R}_{ij} + \frac{2}{n}\tilde{r}\tilde{g}_{ij}$ *of* $\tilde{X} \doteqdot X(\tilde{g})$ *is given by*

$$\frac{\partial \tilde{X}}{\partial t} = \tilde{\Delta}\tilde{X} + \tilde{Y} + k\frac{2}{n}\tilde{r}\tilde{X}. \tag{3.46}$$

REMARK 3.32. *The above lemma also holds when the equalities in (3.45) and (3.46) are replaced by inequalities going the same way.*

EXERCISE 3.33. *Show that if* $[0, \tilde{T})$ *is the maximal time interval of existence of the normalized Ricci flow, then* $\int_0^{\tilde{T}} \tilde{r}\left(\tilde{t}\right) d\tilde{t} = \infty$.

Each of the following estimates represents in some way the fact that under the normalized Ricci flow the metrics converge to constant curvature exponentially fast. The order in which they are stated reflects a natural order in which they are proved.

LEMMA 3.34 (Estimates for the normalized RF). *If* $\left(M^3, g(0)\right)$ *is a closed* 3*-manifold with positive Ricci curvature, then under the normalized Ricci flow we have the following estimates. Let* $[0, \tilde{T})$ *denote the maximal time interval of existence of the normalized Ricci flow. There exist constants* $C < \infty$ *and* $\delta > 0$ *such that*

(1)
$$\lim_{\tilde{t} \to \tilde{T}} \frac{\tilde{R}_{\max}\left(\tilde{t}\right)}{\tilde{R}_{\min}\left(\tilde{t}\right)} = 1$$

(2)
$$\widetilde{\mathrm{Rc}} \geq \delta \tilde{R}\tilde{g}$$

(3)
$$\tilde{R}_{\max}\left(\tilde{t}\right) \leq C$$

(4)
$$\tilde{T} = \infty$$

(5)
$$\lim_{\tilde{t} \to \infty} \left(\max_{\tilde{x} \in M^3} \frac{\left|\widetilde{\mathrm{Rc}} - \frac{1}{3}\tilde{R}\tilde{g}\right|^2}{\tilde{R}^2} \left(\tilde{x}, \tilde{t}\right) \right) = 0$$

(6)
$$\tilde{R}_{\min}\left(\tilde{t}\right) \geq \frac{1}{C}$$

and hence $\operatorname{diam}\left(\tilde{g}(t)\right) \leq C$

(7)
$$\left|\widetilde{\mathrm{Rc}} - \frac{1}{3}\tilde{R}\tilde{g}\right| \leq Ce^{-\delta \tilde{t}}$$

(8)
$$\tilde{R}_{\max}\left(\tilde{t}\right) - \tilde{R}_{\min}\left(\tilde{t}\right) \leq Ce^{-\delta \tilde{t}}$$

(9)

$$\left|\widetilde{\mathrm{Rc}} - \frac{1}{3}\tilde{r}\tilde{g}\right| \leq Ce^{-\delta\tilde{t}} \tag{3.47}$$

(10)

$$\left|\tilde{\nabla}^k \widetilde{\mathrm{Rc}}\right| \leq Ce^{-\delta\tilde{t}} \tag{3.48}$$

for all $k \in \mathbb{N}$.

PROOF. (*Sketch.*) Parts (1), (2) and (5) follow from the corresponding estimates for the unnormalized Ricci flow since the inequalities are scale-invariant.

Part (3): Since $\tilde{R}_{ij} \geq 0$, we have $\text{const} = \mathrm{Vol}\left(\tilde{g}\left(\tilde{t}\right)\right) \leq C \,\mathrm{diam}\left(\tilde{g}\left(\tilde{t}\right)\right)^3$ for a universal constant C. Now since $\widetilde{\mathrm{Rc}} \geq \varepsilon \tilde{R}_{\max}\tilde{g}$ for some $\varepsilon > 0$ (combine (1) and (2)), by Myers' Theorem, we have

$$\mathrm{diam}\left(\tilde{g}\left(\tilde{t}\right)\right) \leq C\tilde{R}_{\max}\left(\tilde{t}\right)^{-1/2} \tag{3.49}$$

and we conclude $\tilde{R}_{\max}\left(\tilde{t}\right) \leq C$.

Part (4) we leave as an exercise using $\int_0^{\tilde{T}} \tilde{r}\left(\tilde{t}\right) d\tilde{t} = \infty$.

Part (6): By Klingenberg's injectivity radius estimate, replacing $\left(M^3, \tilde{g}\left(\tilde{t}\right)\right)$ by their universal covering Riemannian manifolds $\left(\tilde{M}^3, \tilde{\tilde{g}}\left(\tilde{t}\right)\right)$, we have

$$\mathrm{inj}\left(\tilde{\tilde{g}}\left(\tilde{t}\right)\right) \geq \varepsilon \tilde{R}_{\max}\left(\tilde{t}\right)^{-1/2}$$

for some universal constant $\varepsilon > 0$. Since $\mathrm{sect}\left(\tilde{\tilde{g}}\left(\tilde{t}\right)\right) \leq C\tilde{R}_{\max}\left(\tilde{t}\right)$, this implies $\mathrm{Vol}\left(\tilde{\tilde{g}}\left(\tilde{t}\right)\right) \geq \varepsilon \tilde{R}_{\max}\left(\tilde{t}\right)^{-3/2}$ for some other constant $\varepsilon > 0$. Hence we have

$$\text{const} = \mathrm{Vol}\left(\tilde{g}\left(\tilde{t}\right)\right) \geq \delta \tilde{R}_{\max}\left(\tilde{t}\right)^{-3/2}$$

where $\delta > 0$ depends also on $\left|\pi_1\left(M^3\right)\right| < \infty$. Hence $\tilde{R}_{\max}\left(\tilde{t}\right) \geq \frac{1}{C}$ and the same estimate holds for $\tilde{R}_{\min}\left(\tilde{t}\right)$ by (1). Now by (3.49) we also have a uniform upper bound for the diameter of $\tilde{g}\left(\tilde{t}\right)$.

Part (7): Let

$$\tilde{f} \doteqdot \frac{\left|\widetilde{\mathrm{Rc}} - \frac{1}{3}\tilde{R}\tilde{g}\right|^2}{\tilde{R}^2}.$$

$\tilde{f}$ satisfies the same equation as for its counterpart $f \doteqdot \left|\mathrm{Rc} - \frac{1}{3}Rg\right|^2 \big/ R^2$ for the unnormalized flow. This equation is the following (see Lemma 10.5 of [**255**])

$$\frac{\partial f}{\partial t} = \Delta f + 2\left\langle \nabla \log R, \nabla f\right\rangle - \frac{2}{R^4}\left|R\nabla_i R_{jk} - \nabla_i R R_{jk}\right|^2 + 4P,$$

where

$$P \doteqdot \frac{1}{R^3}\left(\frac{5}{2}R^2\left|\mathrm{Rc}\right|^2 - 2R\,\mathrm{Trace}_g\left(\mathrm{Rc}^3\right) - \frac{1}{2}R^4 - \left|\mathrm{Rc}\right|^4\right)$$

This is actually the same P as in (8.74), where now $v_{ij} = R_{ij}$ and $\rho = 0$. Note that when $\mathrm{Rc} = \frac{1}{3}Rg$, we have $P = 0$. One can show that if $R_{ij} \geq \varepsilon R g_{ij}$, where $R > 0$ and $\varepsilon \geq 0$, then

$$P \leq -\varepsilon^2 \frac{\left|\mathrm{Rc}\right|^2\left|\mathrm{Rc} - \frac{1}{3}Rg\right|^2}{R^3};$$

see Lemma 10.7 of [**255**] for details (caveat: our P differs from Hamilton's P by a sign and a factor of R^3.) Hence we have

$$\begin{aligned}\frac{\partial \tilde{f}}{\partial \tilde{t}} &\leq \tilde{\Delta}\tilde{f} + 2\left\langle \tilde{\nabla}\log \tilde{R}, \tilde{\nabla} f\right\rangle - 4\varepsilon^2\frac{\left|\widetilde{\mathrm{Rc}}\right|^2\left|\widetilde{\mathrm{Rc}} - \frac{1}{3}\tilde{R}\tilde{g}\right|^2}{\tilde{R}^3} \\ &\leq \tilde{\Delta}\tilde{f} + 2\left\langle \tilde{\nabla}\log \tilde{R}, \tilde{\nabla} f\right\rangle - \delta\left(\tilde{R}_{\min}\right)\tilde{f}. \qquad (3.50)\end{aligned}$$

The desired exponential decay estimate for $\tilde{f}$ now follows from the maximum principle to (3.50) using $\tilde{R}_{\min} \geq \frac{1}{C} > 0$.

Part (8): We go back to (3.38) and (3.39). Adding these two equations implies that

$$\psi \doteqdot \frac{\left|\nabla R\right|^2}{R} + 168\left(\left|\mathrm{Rc}\right|^2 - \frac{1}{3}R^2\right)$$

satisfies, under the unnormalized Ricci flow:

$$\left(\frac{\partial}{\partial t} - \Delta\right)\psi \leq 672R\left(\left|\mathrm{Rc}\right|^2 - \frac{1}{3}R^2\right).$$

Hence, for the normalized Ricci flow, the corresponding quantity $\tilde{\psi}$ satisfies

$$\left(\frac{\partial}{\partial \tilde{t}} - \tilde{\Delta}\right)\tilde{\psi} \leq Ce^{-\delta\tilde{t}} - \frac{4}{3}\tilde{r}\tilde{\psi}.$$

where we used $672\tilde{R}\left(\left|\widetilde{\mathrm{Rc}}\right|^2 - \frac{1}{3}\tilde{R}^2\right) \leq Ce^{-\delta\tilde{t}}$. Since $\tilde{r} \geq \delta$ for some $\delta > 0$, we can conclude that

$$\left(\frac{\partial}{\partial \tilde{t}} - \tilde{\Delta}\right)\left(e^{\delta\tilde{t}}\tilde{\psi} - C\tilde{t}\right) \leq 0$$

and hence $\tilde{\psi} \leq Ce^{-\delta\tilde{t}}\left(1 + \tilde{t}\right)$. This gives us the gradient estimate $\left|\tilde{\nabla}\tilde{R}\right| \leq Ce^{-\delta\tilde{t}}$. Since the diameters of $\tilde{g}\left(\tilde{t}\right)$ are uniformly bounded, we obtain (8) by integrating the gradient estimate along minimal geodesics.

Part (9) follows from (7) and (8).

We refer the reader to [**255**], Theorem 17.6, for the proof of part (10). This requires some interpolation estimates for the L^p-norms of the derivatives of the Ricci tensor. □

From the above lemma and the fact that we can estimate the derivatives of the metrics in terms of the estimates for the derivatives of the Ricci tensor, one can complete the proof of Theorem 3.6; see [**143**] for details. In particular, by (3.47) and Lemma 5.9, there exists a constant $C < \infty$ such that

$$\frac{1}{C}\tilde{g}(0) \leq \tilde{g}\left(\tilde{t}\right) \leq C\tilde{g}(0)$$

for all $\tilde{t} \in [0, \infty)$, and the metrics $\tilde{g}\left(\tilde{t}\right)$ converge uniformly on compact sets to a continuous metric $\tilde{g}(\infty)$ as $\tilde{t} \to \infty$. The estimates (3.48) imply the exponential convergence in each C^k-norm of $\tilde{g}\left(\tilde{t}\right)$ to $\tilde{g}(\infty)$.

8. Notes and commentary

For the details of those details of the proof of Theorem 3.6 which we do not present, we refer the reader to the original paper [**255**] or the book [**153**].

§**3**. A Riemannian manifold has **2-positive curvature operator** if

$$\lambda_1(\mathrm{Rm}) + \lambda_2(\mathrm{Rm}) > 0.$$

That is, the sum of the lowest two eigenvalues of Rm is positive at every point. H. Chen [**119**] has shown that if $(M^n, g(0))$ is a closed Riemannian manifold with 2-positive curvature operator, then under the Ricci flow $g(t)$ has 2-positive curvature operator for all $t > 0$.

§**4**. An interesting question related to (3.29).

PROBLEM 3.35. *Suppose that* $\left(M^3, g\right)$ *is a complete* 3*-manifold with positive Ricci curvature and* $\mathrm{Rc} \geq \varepsilon R g$ *for some* $\varepsilon > 0$*. Must* M^3 *be compact? If we replace positive by nonnegative above, must* M^3 *be flat?*

If $\left(M^3, g\right)$ has bounded nonnegative sectional curvature and $\mathrm{Rc} \geq \varepsilon R g$ for some $\varepsilon > 0$, then Chen and Zhu [**113**] proved that M^3 must be flat. A related question is the following (see Chapter 8 for more on differential Harnack inequalities).

PROBLEM 3.36. *If* $\left(M^3, g(t)\right)$ *is a complete solution to the Ricci flow on a* 3*-manifold with bounded nonnegative Ricci curvature at each time, can one prove a trace differential Harnack inequality? One could hope for an inequality similar to (8.47).*

Note that a result related to the first problem above, due to Hamilton [**265**], is the following.

THEOREM 3.37. *If* $M^n \subset \mathbb{R}^{n+1}$ *is a* C^∞ *complete, strictly convex hypersurface with* $h_{ij} \geq \varepsilon H g_{ij}$ *for some* $\varepsilon > 0$*, then* M^n *is compact.*

Analogous to the second problem posed above is the following.

PROBLEM 3.38. *Does there exist a Harnack inequality for solutions to the mean curvature flow with nonnegative mean curvature and bounded second fundamental form at each time, can one prove a differential Harnack inequality? Here one hopes for an inequality similar to (8.78).*

§**10**. There is a nice presentation of the Ricci flow on homogeneous manifolds in [**319**], which we have partially followed.

Comparison with mean curvature flow (MCF). It is interesting to compare the Ricci flow with the MCF; scattered throughout the rest of the book we shall see analogies and relations between the two flows. Let $X_t : M^{n-1} \to \mathbb{R}^n$, $t \in [0,T)$, be a solution to the **mean curvature flow**:

$$\frac{\partial X}{\partial t}(p,t) = -H(p,t)\,\nu(p,t), \quad p \in M^{n-1},\ t \in [0,T), \tag{3.51}$$

where H is the mean curvature and ν is the unit outward normal. This is the gradient flow for the volume functional.

LEMMA 3.39 (Basic evolutions under MCF, [**286**]). *We have the following evolution equations for the induced metric g_{ij}, second fundamental form h_{ij}, and H :*

$$\begin{aligned}
\frac{\partial}{\partial t} g_{ij} &= -2Hh_{ij} \\
\frac{\partial}{\partial t} h_{ij} &= \Delta h_{ij} - 2Hh_{ik}h_{kj} + |h|^2 h_{ij} \\
\frac{\partial}{\partial t} H &= \Delta H + |h|^2 H.
\end{aligned}$$

PROOF. We leave this as an exercise using the formulas $H = g^{ij}h_{ij}$,

$$\begin{aligned}
g_{ij} &= \left\langle \frac{\partial X}{\partial x^i}, \frac{\partial X}{\partial x^j} \right\rangle \\
h_{ij} &= \left\langle \frac{\partial X}{\partial x^i}, \frac{\partial \nu}{\partial x^j} \right\rangle = -\left\langle \frac{\partial^2 X}{\partial x^i \partial x^j}, \nu \right\rangle.
\end{aligned}$$

While carrying out the computation, keep in mind that the inner product of a tangential vector with a normal vector is zero. □

EXERCISE 3.40. *Suppose that we have the flow $\frac{\partial x}{\partial t} = -\beta\nu$, where β is some function. Compute the evolution equations for g_{ij}, h_{ij} and H.*

From Lemma 3.39 one can show (see [**286**]) using the maximum principle for tensors that if $H \geq 0$, then the inequalities

$$\alpha H g_{ij} \leq h_{ij} \leq \beta H g_{ij}$$

are preserved under the MCF. Compare this with the 'Ricci pinching is preserved' estimate (3.29). Hamilton [**265**] proved that if a complete hypersurface in euclidean space satisfies $h_{ij} \geq \varepsilon H g_{ij}$, where $\varepsilon > 0$ and $H > 0$, then the hypersurface is compact. Analogously, Chen and Zhu [**113**] proved that if (M^3, g) is a complete Riemannian 3-manifold with bounded nonnegative sectional curvature and $\mathrm{Rc} \geq \varepsilon R$ with $\varepsilon > 0$, then M^3 is either compact or flat.

CONJECTURE 3.41 (Hamilton). *If (M^3, g) is a complete Riemannian 3-manifold with $\mathrm{Rc} \geq \varepsilon R$, where $R > 0$ and $\varepsilon > 0$, then M^3 is compact.*

The Codazzi equations $\nabla_i h_{jk} = \nabla_j h_{ik}$ imply that one can improve the estimate $|\nabla_i h_{jk}|^2 \geq \frac{1}{n-1} |\nabla_i H|^2$ to

$$|\nabla_i h_{jk}|^2 \geq \frac{3}{n+1} |\nabla_i H|^2 \tag{3.52}$$

(this is an improvement only when $n \geq 3$); see [**286**] for details. Compare this with (3.37).

Remark 3.42. *If the hypersurface M^{n-1} is totally umbillic, so that $h = \frac{H}{n-1} g$, then from (3.52) we have $\frac{1}{n-1} |\nabla H|^2 \geq \frac{3}{n+1} |\nabla H|^2$. When $n \geq 3$ this implies $|\nabla H| = 0$. (Compare with Exercise 1.170.)*

Huisken's pinching theorem says that

$$\left| h_{ij} - \frac{1}{n} H g_{ij} \right| \leq C H^{1-\delta}. \tag{3.53}$$

Compare with the 'Ricci pinching improves' estimate (3.30). Pointwise estimates are not sufficient to obtain this since, under the ODE corresponding to the PDE for h_{ij}, the pinching is preserved but not improved. In [**286**] an iteration argument is used to obtain (3.53).

CHAPTER 4

Ricci solitons and other special solutions

In the study of the singularities which exist or form for solutions of partial differential equations including those which arise in geometry, a fundamental notion is that of rescaling and applying monotonicity formula to obtain self-similar solutions which model the solutions near the singularities. Such techniques have been successfully applied to the study of minimal surfaces, harmonic maps, Yang-Mills connections, and solutions of nonlinear heat and Schrödinger equations to name a few. In the field of geometric evolution equations, the singularity models which arise are usually ancient solutions, where the solution exists all the way back to time minus infinity. Among such long existing solutions are the self-similar solutions, which in Ricci flow are called Ricci solitons. In this chapter we shall study some properties and examples of such solutions with a special emphasis on dimension two where the solutions are explicit. We also study other long existing solutions such as the Rosenau solution in §6 and homogeneous solutions in §10. The examples in this chapter are very useful to keep in mind when studying singularity formation of solutions of the Ricci flow.

1. Types of long existing solutions

From taking limits of dilations of singularities we shall obtain long existing solutions. Recall that a solution $(M^n, g(t))$ to the Ricci flow is called an **ancient solution** if it is defined on an interval of the form $(-\infty, \omega)$, where $\omega \in \mathbb{R}^+ \cup \{\infty\}$. We say that a solution $(M^n, g(t))$ to the Ricci flow is an **immortal solution** if it is defined on a time interval $\alpha < t < \infty$. Finally, if $(M^n, g(t))$ is defined for all $-\infty < t < \infty$, then we call it an **eternal solution**.

2. Gradient Ricci solitons

We say that a quadruple $(M^n, g_0, f_0, \varepsilon)$, where (M^n, g_0) is a Riemannian manifold, $f_0 : M^n \to \mathbb{R}$ is a function and $\varepsilon \in \mathbb{R}$, is a **gradient Ricci soliton** if

$$\mathrm{Rc}(g_0) + \nabla^{g_0}\nabla^{g_0} f_0 + \frac{\varepsilon}{2} g_0 = 0. \tag{4.1}$$

For reasons we shall see below, we say that g_0 is **expanding**, **shrinking**, or **steady**, if $\varepsilon > 0$, $\varepsilon < 0$, or $\varepsilon = 0$, respectively. We say that the gradient soliton is **complete** if (M^n, g_0) is complete and the vector field $\mathrm{grad}_{g_0} f_0$ is complete.

The following result gives a canonical form for the associated time-dependent version of a gradient Ricci soliton.

THEOREM 4.1 (Gradient Ricci solitons). *If $(M^n, g_0, f_0, \varepsilon)$ is a complete gradient Ricci soliton, then there exists a solution $g(t)$ of the Ricci flow with $g(0) = g_0$, diffeomorphisms $\varphi(t)$ with $\varphi(0) = \mathrm{id}_{M^n}$, functions $f(t)$ with $f(0) = f_0$ defined for all t with*

$$\tau(t) \doteqdot \varepsilon t + 1 > 0, \tag{4.2}$$

such that

(1) *$\varphi(t) : M^n \to M^n$ is the 1-parameter family of diffeomorphisms generated by $X(t) \doteqdot \frac{1}{\tau(t)} \operatorname{grad}_{g_0} f_0$. That is:*

$$\frac{\partial}{\partial t}\varphi(t)(x) = \frac{1}{\tau(t)} (\operatorname{grad}_{g_0} f_0)(\varphi(t)(x)).$$

(2) *$g(t)$ is the pull back by $\varphi(t)$ of g_0 up to the scale factor $\tau(t)$:*

$$g(t) = \tau(t)\, \varphi(t)^* g_0, \tag{4.3}$$

(3) *$f(t)$ is the pull back by $\varphi(t)$ of f_0 :*

$$f(t) = f_0 \circ \varphi(t) = \varphi(t)^* (f_0), \tag{4.4}$$

Moreover,

$$\boxed{\operatorname{Rc}(g(t)) + \nabla^{g(t)}\nabla^{g(t)} f(t) + \frac{\varepsilon}{2\tau} g(t) = 0} \tag{4.5}$$

and

$$\boxed{\frac{\partial f}{\partial t}(t) = \left|\operatorname{grad}_{g(t)} f(t)\right|^2_{g(t)}.} \tag{4.6}$$

Caveat: Unless $\varepsilon = 0$, i.e., $\tau(t) \equiv 1$, the 1-parameter family $\varphi(t)$ is not a group.

PROOF. Define $\tau(t) = \varepsilon t + 1$. Since the vector field $\operatorname{grad}_{g_0} f_0$ is complete, there exists a parameter family of diffeomorphisms $\varphi(t) : M^n \to M^n$ generated by the vector fields $\frac{1}{\tau(t)} \operatorname{grad}_{g_0} f_0$ defined for all t such that $\tau(t) > 0$. Then define $f(t) = f_0 \circ \varphi(t)$ and $g(t) = \tau(t)\, \varphi(t)^* g_0$. We compute

$$\left.\frac{\partial}{\partial t}\right|_{t=t_0} g(t) = \frac{\varepsilon}{\tau(t_0)} g(t_0) + \tau(t_0) \left.\frac{\partial}{\partial t}\right|_{t=t_0} (\varphi(t)^* g_0).$$

Using Remark 1.22 we have

$$\begin{aligned} \tau(t_0) \left.\frac{\partial}{\partial t}\right|_{t=t_0} (\varphi(t)^* g_0) &= \tau(t_0)\, \mathcal{L}_{\left(\varphi(t_0)^{-1}\right)_* \frac{\partial}{\partial t}\big|_{t=t_0} \varphi(t)} \varphi(t_0)^* g_0 \\ &= \mathcal{L}_{\operatorname{grad}_{g(t_0)} f(t_0)} g(t_0) \end{aligned}$$

which holds since

$$\left.\frac{\partial}{\partial t}\right|_{t=t_0} \varphi(t) = \frac{1}{\tau(t_0)} \operatorname{grad}_{g_0} f_0 = \varphi(t_0)_* \left(\operatorname{grad}_{g(t_0)} f(t_0)\right).$$

Hence (evaluating at t instead of t_0)

$$\frac{\partial}{\partial t} g(t) = \frac{\varepsilon}{\tau(t)} g(t) + \mathcal{L}_{\operatorname{grad}_{g(t)} f(t)} g(t).$$

Now using Exercise 1.21 we find that

$$\begin{aligned} -2\operatorname{Rc}(g(t)) &= \varphi(t)^* (-2\operatorname{Rc}(g_0)) = \varphi(t)^* \left(\varepsilon g_0 + \mathcal{L}_{\operatorname{grad}_{g_0} f_0} g_0\right) \\ &= \frac{\varepsilon}{\tau(t)} g(t) + \mathcal{L}_{\operatorname{grad}_{g(t)} f(t)} g(t). \end{aligned}$$

Hence

$$\frac{\partial}{\partial t} g(t) = \frac{\varepsilon}{\tau} g(t) + \mathcal{L}_{\operatorname{grad}_{g(t)} f(t)} g(t) = -2\operatorname{Rc}(g(t)). \tag{4.7}$$

Finally we calculate

$$\begin{aligned} \frac{\partial f}{\partial t}(x,t) &= \left(\frac{\partial}{\partial t}\varphi(t)\right)(f_0)(x) = \frac{1}{\tau(t)} \left|\operatorname{grad}_{g_0} f_0\right|^2 (\varphi(t)(x)) \\ &= \left|\operatorname{grad}_{g(t)} f(t)\right|^2_{g(t)} (x). \end{aligned}$$

□

Note that, as advertised above, $g(t)$ is expanding, shrinking, or steady, if $\varepsilon > 0$, $\varepsilon < 0$, or $\varepsilon = 0$, respectively. In local coordinates we may write (4.7) as

$$\frac{\partial}{\partial t} g_{ij} = -2R_{ij} \tag{4.8a}$$

$$S^{\varepsilon}_{ij} \doteqdot R_{ij} + \nabla_i \nabla_j f + \frac{\varepsilon}{2\tau} g_{ij} = 0. \tag{4.8b}$$

REMARK 4.2. *If $\varepsilon = 0$, then $g(t)$ is defined for all $t \in (-\infty, \infty)$; if $\varepsilon > 0$, then $g(t)$ is defined for $t \in (-1/\varepsilon, \infty)$; and if $\varepsilon < 0$, then $g(t)$ is defined for $t \in (-\infty, 1/|\varepsilon|)$. For this reason we say that the solution is* ***eternal****,* ***immortal*** *or* ***ancient****, respectively.*

By dilating the solution, except when $g(t)$ is steady, we can choose $|\varepsilon|$ to be any positive real number. The above theorem, by requiring (4.2), puts the gradient Ricci soliton in canonical form. In a class of solutions where uniqueness of the initial value problem holds, by adjusting the diffeomorphisms, this is always possible (see Chapter 1 of [**143**]).

3. Gaussian soliton

Euclidean space $\mathbb{R}^n$ with the standard flat metric $g(t) \equiv g_{\text{can}}$ may seem like an uninteresting solution to the Ricci flow since it is stationary. However, one of the reasons it is interesting is that the metrics are invariant under scaling: for any constant $c > 0$, cg_{can} is isometric to g_{can}. In particular, $cg_{\text{can}} = \varphi^* g_{\text{can}}$, where $\varphi = \sqrt{c}\,\mathrm{id}_{\mathbb{R}^n}$. Because of this, we may think of $(\mathbb{R}^n, g_{\text{can}})$ not only as a steady gradient Ricci soliton but also as either an expanding or a shrinking gradient Ricci soliton.

Let $\tau : \mathcal{I} \to (0, \infty)$ be a smooth function on an interval and define $\varphi(t) \doteqdot \tau(t)^{-1/2}\,\mathrm{id}_{\mathbb{R}^n}$ for $t \in \mathcal{I}$. Then

$$g(t) \equiv g_{\text{can}} = \tau(t)\,\varphi(t)^*\,g(0)\,. \tag{4.9}$$

When viewed this way, we call the stationary flat solution g_{can} the **Gaussian soliton**. Choose $\tau(t) = \varepsilon t + 1$ where $\varepsilon \in \mathbb{R}$, as in (4.2), so that

$$\varphi(t) = \tau(t)^{-1/2}\,\mathrm{id}_{\mathbb{R}^n}$$

is a 1-parameter family of diffeomorphisms. If we define

$$f(x,t) \doteqdot -\frac{\varepsilon\,|x|^2}{4\tau(t)}, \tag{4.10}$$

then $f(t) = f(0) \circ \varphi(t)$, $\operatorname{grad}_{g(0)} f(0) = -\frac{\varepsilon}{2} x$ and

$$\frac{d}{dt}\varphi(t)(x) = -\frac{\varepsilon}{2}\tau(t)^{-3/2}\,x = \frac{1}{\tau(t)}\left(\operatorname{grad}_{g(0)} f(0)\right)(\varphi(t)(x))\,.$$

At time t we calculate $(\mathrm{Rc}(g(t)) \equiv 0)$

$$2\,\mathrm{Rc}(g(t)) + \frac{\varepsilon}{\tau(t)} g(t) + \mathcal{L}_{\operatorname{grad}_{g(t)} f(t)} g(t) = \frac{\varepsilon}{\tau(t)} g_{\text{can}} - \frac{\varepsilon}{4\tau(t)} \mathcal{L}_{\operatorname{grad}_{g_{\text{can}}} |x|^2} g_{\text{can}} = 0$$

which agrees with what we know from the proof of Theorem 4.1.

REMARK 4.3. *Considering $\mathbb{R}^n$ as a shrinking soliton, so that $\varepsilon = -1$, we have*

$$f(x,t) = \frac{|x|^2}{4\tau}. \tag{4.11}$$

For a further discussion of the Gaussian soliton and its relation to Perelman's entropy and Harnack, see Volume 2.

4. Cylinder shrinking soliton

Consider the product of the shrinking sphere with a line: $\left(S^{n-1} \times \mathbb{R}, g(t)\right)$, $t \in (-\infty, 0)$, $n \geq 3$, where

$$g(t) = 2(n-2)\,|t|\,g_{S^{n-1}} + dr^2.$$

Its Ricci tensor is given by

$$\mathrm{Rc}(g(t)) = (n-2)\,g_{S^{n-1}} = \frac{1}{2|t|} g(t) - \frac{1}{2|t|} dr^2.$$

If we let

$$f(\theta, r, t) \doteqdot \frac{r^2}{4|t|}, \quad \theta \in S^{n-1},\ r \in \mathbb{R},\ t < 0, \tag{4.12}$$

we then have

$$\operatorname{Rc}(g(t)) + \nabla\nabla f(t) + \frac{1}{2t} g(t) = 0.$$

Hence $g(t)$ is a shrinking gradient Ricci soliton. Since the shrinking cylinder models neck pinches, it is a very important example.

EXERCISE 4.4. *Show that* $\left(S^n \times \mathbb{R}^k, g(t)\right)$, $t \in (-\infty, 0)$, $n \geq 2$, *where*

$$g(t) = 2(n-1)|t|\, g_{S^{n-1}} + g_{\mathbb{E}^k},$$

is a shrinking gradient soliton with

$$f(\theta, x, t) \doteqdot \frac{|x|^2}{4|t|}, \quad \theta \in S^{n-1},\ x \in \mathbb{R},\ t < 0.$$

In a sense, the steady (stationary) euclidean metric is turned into a shrinking soliton by taking the product with a shrinking sphere.

Note that in the above example

$$\nabla f = \frac{x}{2|t|}, \qquad |\nabla f|^2 = \frac{|x|^2}{4t^2}.$$

In particular, ∇f is radial and pointing outward. In contrast, by (4.16), for the cigar soliton $\nabla f = -2x$ is pointing inward.

The cylinder soliton is important since in dimension 3 it models neck-pinching. Note the form of the potential function in (4.12) motivates some of the estimates in the proof in Volume 2 on the nonexistence of noncompact 3-dimensional shrinking solitons with positive sectional curvature.

5. Cigar steady soliton

Hamilton's **cigar soliton** is the complete Riemannian surface $\left(\mathbb{R}^2, g_\Sigma\right)$, where

$$g_\Sigma \doteqdot \frac{dx^2 + dy^2}{1 + x^2 + y^2}. \tag{4.13}$$

where $dx^2 \doteqdot dx \otimes dx$. As a solution to the Ricci flow, its time-dependent version is:

$$\boxed{g_\Sigma(t) \doteqdot \frac{dx^2 + dy^2}{e^{4t} + x^2 + y^2}.} \tag{4.14}$$

To see that $g_\Sigma(t)$ is a solution to the Ricci flow, we leave it to the reader to check that $u(x, y, t) \doteqdot -\log\left(e^{4t} + x^2 + y^2\right)$ satisfies equation (**??**) with $h = dx^2 + dy^2$: $\frac{\partial u}{\partial t} = \Delta \log u$, where $\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ is the euclidean Laplacian. From the change of variables $\tilde{x} = e^{-2t}x$ and $\tilde{y} = e^{-2t}y$, we see that $g_\Sigma(t) = \frac{d\tilde{x}^2 + d\tilde{y}^2}{1 + \tilde{x}^2 + \tilde{y}^2}$ is isometric to $g_\Sigma = g_\Sigma(0)$. That is, if we define the 1-parameter group of (conformal) diffeomorphisms $\varphi_t : \mathbb{R}^2 \to \mathbb{R}^2$ by $\varphi_t(x,y) \doteqdot \left(e^{-2t}x, e^{-2t}y\right)$, then

$$g_\Sigma(t) = \varphi_t^* g_\Sigma(0).$$

Hence $g_\Sigma(t)$ is a steady Ricci soliton. Geometrically a domain $\Omega \subset \mathbb{R}^2$ with respect to the metric $g_\Sigma(0)$ corresponds to the domain $\varphi_t^{-1}(\Omega) = e^{2t}\Omega \subset \mathbb{R}^2$ with respect to the metric $g_\Sigma(t)$. That is, $(\Omega, g_\Sigma(0))$ is isometric to $\left(e^{2t}\Omega, g_\Sigma(t)\right)$. The vector field generating the 1-parameter group φ_t is $(-2x,-2y) = \operatorname{grad}_{g_\Sigma} f$, where

$$f(x,y) \doteqdot -\log\left(1+x^2+y^2\right). \tag{4.15}$$

Note that

$$\operatorname{grad}_{g_\Sigma(t)} f(t) = (-2x,-2y) \tag{4.16}$$

is independent of time, where

$$f(x,y,t) \doteqdot -\log\left(1+e^{-4t}\left(x^2+y^2\right)\right) = (f\circ\varphi_t)(x,y).$$

This manifold is also known in the physics literature as **Witten's black hole**. As we shall see below, this is a Ricci soliton metric. In polar coordinates, we may rewrite the cigar metric as

$$g_\Sigma = \frac{dr^2 + r^2\, d\theta^2}{1+r^2}. \tag{4.17}$$

If we define the new radial distance variable:

$$s \doteqdot \operatorname{arcsinh} r = \log\left(r + \sqrt{1+r^2}\right),$$

then we may rewrite g_Σ as

$$g_\Sigma = ds^2 + \tanh^2 s\, d\theta^2. \tag{4.18}$$

In general, if a metric takes the form $g = ds^2 + \phi(s)^2 d\theta^2$, then its scalar curvature is $R(s,\theta) = -2\frac{\phi''(s)}{\phi(s)}$. We can see this from the following **moving frames** calculation (see [**282**] for a nice exposition of this technique). Let

$$\omega^1 = ds, \quad \omega^2 = \phi(s)\, d\theta.$$

The connection 1-forms ω_i^j, defined by $\nabla_X e_i = \omega_i^j(X)\, e_j$, and the curvature 2-forms Ω_i^j, defined by $\operatorname{Rm}(X,Y)\, e_i = \Omega_i^j(X,Y)\, e_j$, satisfy the **Cartan structure equations** (using $n=2$ in the equation for Ω_i^j)

$$d\omega^i = \omega^j \wedge \omega_j^i, \quad \Omega_i^j = d\omega_i^j$$

From this we compute

$$\omega_1^2 = \phi'(s)\, d\theta, \quad \Omega_1^2 = \frac{\phi''(s)}{\phi(s)}\omega^1 \wedge \omega^2,$$

and $K = \Omega_1^2(e_2,e_1) = -\frac{\phi''(s)}{\phi(s)}$. Thus the scalar curvature of g_Σ is

$$R_\Sigma = 4\operatorname{sech}^2 s = \frac{4}{1+r^2}.$$

From (4.15) we have

$$f(s) = -2\log(\cosh s).$$

We then have

$$R_{ij} + \nabla_i \nabla_j f = 0 \tag{4.19}$$

which is the infinitesimal version of the steady Ricci soliton equation for $g_\Sigma(t)$. We can also verify this directly as follows. The Hessian of f is given by:

$$\nabla_{e_i} \nabla_{e_j} f = e_i(e_j(f)) - \omega_j^k(e_i)\, e_k(f). \tag{4.20}$$

The first term on the RHS is nonzero only if $i = j = 1$ whereas the second term on the RHS is nonzero only if $i = j = 2$. Since $\omega_2^1(e_2) = -\frac{\phi'(s)}{\phi(s)} = -\frac{1}{\sinh s \cosh s}$ and $e_1(f) = -2\tanh s$, we have

$$e_1(e_1(f)) = -\omega_2^1(e_2)\, e_1(f) = -2\,\mathrm{sech}^2 s = -\frac{R}{2}$$

and (4.19) follows.

EXERCISE 4.5 (Cigar form). *By making the change of variables* $r \doteqdot M\cosh^2 s$, *show that the following is another form of the cigar metric:*

$$g = \left(1 - \frac{M}{r}\right) d\theta^2 + \left(1 - \frac{M}{r}\right)^{-1} \frac{dr^2}{4r^2}, \tag{4.21}$$

where $r > M$.

Now consider the cylinder $\mathbb{R} \times S^1(1)$, where $S^1(1)$ denotes the circle of radius 1, and let $x \in \mathbb{R}$ and $\theta \in S^1(1) = \mathbb{R}/2\pi\mathbb{Z}$ be the standard coordinates. Let h be the flat metric $h = dx^2 + d\theta^2$. The **cigar metric on the cylinder** (punctured plane) may be written as

$$g_{\Sigma^2 - 0} = \left(e^{-2x} + 1\right)^{-1} \left(dx^2 + d\theta^2\right). \tag{4.22}$$

Defining $x \doteqdot \frac{1}{2}\log\left(\frac{r}{M} - 1\right) > 0$ so that $1 - \frac{M}{r} = \left(e^{-2x} + 1\right)^{-1}$ and $dx = \frac{dr}{2(r-M)}$, we see that (4.21) implies (4.22). We can also see (4.22) by solving for F and x in the equation

$$F(x)^2 \left(dx^2 + d\theta^2\right) = ds^2 + \tanh^2 s\, d\theta^2$$

where we get $F(x)\, dx = ds$ and $F(x) = \tanh s$ so that $x = \log(\sinh s)$, $s > 0$, and $F(x)^2 = \frac{e^{2x}}{1+e^{2x}} = \left(e^{-2x} + 1\right)^{-1}$ as desired.

REMARK 4.6. *The metric* $g_{\Sigma^2 - 0}$ *is incomplete; however, by taking the 1-point compactification of* $\mathbb{R} \times S^1(1)$ *at the end where* $x \to -\infty$, *we obtain the complete metric cigar metric on* $\mathbb{R}^2$. *Note that* $u(x) \doteqdot \left(e^{-2x} + 1\right)^{-1}$ *satisfies* $\lim_{x \to \infty} u(x) = 1$ *so at that end the metric is asymptotically cylindrical as we know.*

For the proof of the following classification result, see [**143**].

LEMMA 4.7 (Positively curved 2-d gradient steady soliton is a cigar). *If* $(\mathbb{R}^2, g(t))$ *is a steady gradient Ricci soliton conformal to the standard metric on* $\mathbb{R}^2$, *then* $(\mathbb{R}^2, g(t))$ *is either the cigar soliton or the flat metric.*

If (M^2, g) is a complete noncompact Riemannian surface with positive curvature, then (M^2, g) is conformal to the standard metric on $\mathbb{R}^2$ (see Huber [**285**]).

COROLLARY 4.8 (Uniqueness of the cigar). *If* $(M^2, g(t))$ *is a complete steady gradient Ricci soliton with positive curvature, then* $(M^2, g(t))$ *is the cigar soliton.*

In Hamilton's program for the Ricci flow on 3-manifolds, via dimension reduction, the cigar soliton is a potential singularity model. Perelman's monotonicity formula(s) rule out this possibility (see Volume 2.)

6. Rosenau solution

Let $(\mathbb{R}\times S^1(2), h)$ denote the flat cylinder, where $h = dx^2 + d\theta^2$ and $\theta \in S^1(2) = \mathbb{R}/4\pi\mathbb{Z}$. The **Rosenau solution** [**434**] is the solution $g(t) = u(t)\cdot h$ to the Ricci flow defined for $t < 0$ by

$$u(x,t) = \frac{\sinh(-t)}{\cosh x + \cosh t}. \tag{4.23}$$

One computes that its curvature is

$$R[g(t)] = -\frac{\Delta_h \log u}{u} = \frac{\cosh t \cdot \cosh x + 1}{\sinh(-t)(\cosh x + \cosh t)}.$$

From this we easily check that

$$\frac{\partial}{\partial t} u = -Ru$$

so that $g(t)$ is a solution to the Ricci flow. The metrics $g(t)$ defined on $\mathbb{R}\times S^1(2)$ extend to smooth metrics, which we also call $g(t)$, on the 2-sphere S^2, which is obtained by compactifying $\mathbb{R}\times S^1(2)$ by adding two points (we call these two points the north and south poles). We can see this as follows. For x large and negative (similarly when positive), $g(t)$ is asymptotic to a constant multiple $(2\sinh(-t))$ of the metric $e^x(dx^2 + d\theta^2) = 4(d\bar{x}^2 + \bar{x}^2 d\bar{\theta}^2)$, where $\bar{x} = e^{x/2}$ and $\bar{\theta} = \theta/2$; note $\bar{\theta} \in S^1(1) = \mathbb{R}/2\pi\mathbb{Z}$, and when x is large and negative $\bar{x}$ is positive and near 0 (see [**153**], p. 33 for more details).

EXERCISE 4.9. *Show that*

$$\sup_{S^2} R[g(t)] = \coth(-t).$$

In particular $\lim_{t\to-\infty} \sup_{S^2} R[g(t)] = 1$.

EXERCISE 4.10 (Rosenau tends to round at singularity time). *Show that since* $\lim_{t\to 0}\sinh(-t)\,u(x,t) = \frac{1}{\cosh x+1}$, *the limit of* $\sinh(-t)\,g(t)$ *as* $t \to 0$ *is the round* 2*-sphere with scalar curvature* 1 *(radius* $\sqrt{2}$*).* HINT*: use formula* **??**.

We now take a limit of the Rosenau solution as $t \to -\infty$ to see that we can get the cigar soliton as a (backward) limit (we can also get the cylinder). Note that the Cheeger-Gromov type compactness theorem (see Theorem 5.19) yields convergence only after pulling back by diffeomorphisms. Fortunately, in the case of the Rosenau solution, the diffeomorphisms sufficient to obtain convergence are translations. In particular, consider

$$u(x+t,t) = (-\cosh x \coth t - \sinh x - \coth t)^{-1},$$

so that

$$\lim_{t\to-\infty} u(x+t,t) = (\cosh x - \sinh x + 1)^{-1} = \left(e^{-x}+1\right)^{-1}.$$

Let $\phi_t : \mathbb{R}\times S^1(2) \to \mathbb{R}\times S^1(2)$ be defined by $\phi_t(x,\theta) = (x+t,\theta)$. Then $(\phi_t^* g)(x,t) = u(x+t,t)\,h$. Hence

$$\lim_{t\to-\infty} (\phi_t^* g)(x,t) = \left(e^{-x}+1\right)^{-1} h(x). \tag{4.24}$$

Similarly, for the diffeomorphisms $\psi_t(x,\theta) = (x-t,\theta)$ we have

$$\lim_{t\to-\infty} (\psi_t^* g)(x,t) = (e^{x}+1)^{-1} h(x). \tag{4.25}$$

Making the change of variables $\tilde{x} = x/2$ and $\tilde{\theta} = \theta/2$ in (4.24), we have

$$\left(e^{-x}+1\right)^{-1} h(x) = 4\left(e^{-2\tilde{x}}+1\right)^{-1}\left(d\tilde{x}^2 + d\tilde{\theta}^2\right), \tag{4.26}$$

where $\tilde{x} \in \mathbb{R}$ and $\tilde{\theta} \in S^1(1)$. Similarly, we could do the same with (4.25). Thus we obtain the cigar soliton (compare (4.26) with (4.22)) as a backward limit in two essentially different ways, corresponding to dilating about points close enough to either the north or south pole. It is interesting that we did not need to rescale (dilate) to obtain the cigar. Note that we can also obtain the cylinder as a backward limit since $\lim_{t\to-\infty} u(x,t) = 1$, so that

$$\lim_{t\to-\infty} g(x,t) = h(x)$$

for all $x \in \mathbb{R}$. This corresponds to dilating about points close enough to the center circle (equator) $\{(x,\theta) : x = 0\}$, and in particular, dilating about points *on* the center circle as above.

REMARK 4.11. *In the above we are taking the limit of a 1-parameter family of metrics (as* $t \to -\infty$*) to a* single *Ricci soliton metric. We leave it as an exercise for the reader to take the analogous limit of a 1-parameter family of solutions of Ricci flow to a soliton* solution *of the Ricci flow. In the Cheeger-Gromov compactness theorem one takes a limit of a* sequence *of solutions.*

In the dimension reduction of a 3-dimensional singularity model, one obtains an ancient solution on a surface. For singularity models arising from finite time singularities, Perelman's no local collapsing theorem rules out the Rosenau solution. However, for singularity models arising from infinite time singularities, the Rosenau solution has not been ruled out.

7. An expanding soliton

We say that a solution $(M^n, g(t))$ to the Ricci flow is an **immortal solution** if it is defined on a time interval $\alpha < t < \infty$. In dimension 3 there are a number of immortal solutions which are **locally homogeneous** and whose curvatures decay like $1/t$; these have been studied by Isenberg and Jackson [**297**] (see also Chapter 1 of [**153**]; for the study of 4-dimensional homogeneous solutions, see Isenberg-Jackson-Lu [**298**]). Further investigations in this direction, concerning **quasi-convergence**, have been undertaken by Knopf and McLeod [**320**], [**324**].

In dimension 2, there are nontrivial (that is, nonconstant curvature) examples of expanding Ricci solitons (see Appendix A of Gutperle, Headrick, Minwalla and Schomerus [**252**]). In particular, consider rotationally symmetric metrics $g(t)$ on $\mathbb{R}^2$ of the form:

$$g(t) = t\left(F(r)^2\, dr^2 + r^2\, d\theta^2\right), \tag{4.27}$$

where $F : [0,\infty) \to (0,\infty)$ is a positive function to be determined by the expanding Ricci soliton condition, and where $r \in (0,\infty)$ and $\theta \in \mathbb{R}/(2\pi F(0))$. That is, each metric $g(t)$ is first defined on $(0,\infty) \times S^1(F(0)) \cong \mathbb{R}^2 - \{0\}$. We define θ in this range to ensure that the cone angle at the origin is 2π so that $g(t)$ indeed define smooth metrics on $\mathbb{R}^2$ by extending smoothly over the origin ($r \to 0$). Note that these metrics $g(t) = tg(1)$ are homothetically expanding for $t > 0$. One can compute that the Gauss curvatures (1/2 the scalar curvature) are given by

$$K[g(t)] = \frac{1}{t}\frac{F'(r)}{rF(r)^3}. \tag{4.28}$$

Indeed, an orthonormal coframe is given by

$$\omega^1 = \sqrt{t}F(r)\, dr \quad \omega^2 = \sqrt{t}r d\theta,$$

and one has

$$\omega_1^2 = \frac{1}{F(r)} d\theta$$

$$\Omega_1^2 = d\omega_1^2 = -\frac{1}{t}\frac{F'(r)}{rF(r)^3}\omega^1 \wedge \omega^2.$$

Let $X(t)$ be the vector fields on $\mathbb{R}^2$ defined by

$$X(t) \doteqdot \frac{r}{tF(r)}\frac{\partial}{\partial r} = \frac{1}{t}X(1). \tag{4.29}$$

Strictly speaking, $X(t)$ is defined on $\mathbb{R}^2 - \{0\}$; however, it will extend smoothly to $\mathbb{R}^2$. Note that for a radial function $f(r)$, its gradient with respect to $g(t)$ is given by $\operatorname{grad}_{g(t)} f = \frac{1}{tF(r)^2} f'(r) \frac{\partial}{\partial r}$. Hence

$$X(t) = \operatorname{grad}_{g(t)} f \quad \text{where } f'(r) = rF(r). \tag{4.30}$$

We look for F such that $g(t)$ is a solution to the modified Ricci flow

$$\frac{\partial}{\partial t} g(t) = -R_{g(t)} g(t) + \mathcal{L}_{X(t)} g(t). \tag{4.31}$$

The dual 1-forms to $X(t)$ are $X(t)^\flat = rF(r)\, dr = \frac{r}{\sqrt{t}} \omega^1$, which is time-independent. Recall that, in general, from the first structure equations:

$$\left(\nabla_V \omega^j\right)(e_i) = -\omega^j\left(\nabla_V e_i\right) = -\omega_i^j(V)$$

for any vector V, so that $\nabla \omega^j = -\omega_i^j \otimes \omega^i$. Hence, using this and $\omega_2^1 = -\frac{1}{\sqrt{t} r F(r)} \omega^2$, we find that

$$\mathcal{L}_{X(t)} g(t) = 2 \operatorname{Sym}\left(\nabla X(t)^\flat\right) = \frac{2}{\sqrt{t}} \operatorname{Sym}\left(dr \otimes \omega^1 + r \nabla \omega^1\right) = \frac{2}{tF(r)} g(t).$$

Thus (4.31) is equivalent to the equation

$$F'(r) = rF(r)^2 \left(1 - \frac{F(r)}{2}\right). \tag{4.32}$$

(Note that $F(r) \equiv 2$ is the flat euclidean solution, but this is not the one we are interested in.) Now from (4.30) and (4.32)

$$f'(r) = rF(r) = \frac{F'(r)}{F(r)\left(1 - \frac{F(r)}{2}\right)}.$$

Integrating this we have

$$f(r) = -\log\left(\frac{2}{F(r)} - 1\right). \tag{4.33}$$

REMARK 4.12.

(1) *Equation (4.31) is equivalent to the static equation*

$$g(1) = -R_{g(1)} g(1) + \mathcal{L}_{X(1)} g(1).$$

(2) *The metrics $\psi(t)^* g(t)$ satisfy Ricci flow if the 1-parameter family of diffeomorphisms $\psi(t)$ satisfy*

$$\left.\frac{d}{dt}\right|_{t=t_0} \left(\psi(t) \circ \psi^{-1}(t_0)\right) = -\psi(t_0)^* X(t_0).$$

Solving the separable ODE (4.32) we obtain

$$h(r) + \log h(r) = -r^2 + C. \tag{4.34}$$

where $h(r) \doteqdot \frac{2}{F(r)} - 1$. Here we have made the assumption that $0 < F < 2$ so that $h > 0$. Note that by (4.28) and (4.32) we have

$$K[g(t)] = \frac{1}{t}\left(\frac{1}{F(r)} - \frac{1}{2}\right) > 0. \tag{4.35}$$

Recall that the **product log** (or **Lambert-W**) **function** $W : \mathbb{R}^+ \to \mathbb{R}^+$ is the inverse of the function $w(x) = xe^x$. Hence, taking the exponential of both sides of (4.34), we have

$$h(r)\, e^{h(r)} = h(0)\, e^{h(0)} e^{-r^2},$$

so that

$$h(r) = W\left(h(0)\, e^{h(0)-r^2}\right).$$

In terms of F, this equation says:

$$F(r) = \frac{2}{W\left(\left(\frac{2}{F(0)} - 1\right)\exp\left(\frac{2}{F(0)} - 1 - r^2\right)\right) + 1}. \tag{4.36}$$

The **cone angle** of $g(t)$ at infinity (which is independent of t) is

$$\text{ConeAngle} = \frac{2\pi F(0)}{\lim_{r\to\infty} F(r)} = \pi F(0).$$

Hence the range of possible cone angles at infinity is $(0, 2\pi)$ since $F(0) \in (0, 2)$. Now (4.35) and (4.36) imply

$$K[g(t)] = \frac{1}{2t} W\left(\left(\frac{2}{F(0)} - 1\right)\exp\left(\frac{2}{F(0)} - 1 - r^2\right)\right).$$

Finally we note from (4.33) that we have

$$f(r) = -\log h(r) = -\log W\left(h(0)\, e^{h(0)} e^{-r^2}\right).$$

EXERCISE 4.13 (Exponential curvature decay of expanding 2d Ricci soliton). *Show that each of the metrics $g(t)$ is asymptotic at infinity to a flat cone and the curvature decays exponentially as a function of the distance to the origin.* HINT: $\lim_{x\to 0_+} \frac{W(x)}{x} = 1$.

In higher dimensions there are expanding **Kähler–Ricci solitons** on $\mathbb{C}^n$ due to Cao [**73**] (compare also Feldman, Ilmanen and Knopf [**206**].)

Expanding gradient solitons motivate the definition of the matrix Harnack quadratic (see §8.5).

8. Bryant soliton

Let $g_{S^{n-1}}$ denote the standard metric on the unit $(n-1)$-sphere. We search for warped product Ricci solitons on $(0, \infty) \times S^{n-1}$ which extend to Ricci solitons on $\mathbb{R}^n$ by a 1-point compactification of one end. We call the compactifying point the **origin** O. In particular, consider metrics of the form

$$g = dr^2 + \phi(r)^2 g_{S^{n-1}}. \tag{4.37}$$

From $\text{Rc}(g_{S^{n-1}}) = (n-2)\, g_{S^{n-1}}$ and a standard formula for the Ricci tensor of a **warped product metric** (see [**48**], Prop. 9.106), we have

$$\text{Rc}(g) = -(n-1)\frac{\phi''}{\phi}dr^2 + \left((n-2)\left(1-(\phi')^2\right) - \phi\phi''\right) g_{S^{n-1}}.$$

One can also see this from the formulas for the sectional curvatures. The sectional curvature of a plane passing through the radial vector $\frac{\partial}{\partial r}$ is

$$K_{\text{rad}} = -\frac{\phi''(r)}{\phi(r)} \tag{4.38}$$

and the sectional curvature of a plane P_{sph} perpendicular to $\frac{\partial}{\partial r}$ is

$$K_{\text{sph}} = \frac{1-\phi'(r)^2}{\phi(r)^2}. \tag{4.39}$$

We can also calculate that the Hessian of a function f is given by

$$\nabla\nabla f = f''(r)dr^2 + \phi\phi' f' g_{S^{n-1}}.$$

The Ricci soliton equation $\text{Rc}(g) + \nabla\nabla f = 0$ becomes the following system of two second order ODE

$$f'' = (n-1)\frac{\phi''}{\phi}, \qquad \phi\phi' f' = -(n-2)(1-(\phi')^2) + \phi\phi''.$$

Making the substitutions

$$x = \phi', \quad y = \phi f' + (n-1)\phi', \quad dt = \frac{dr}{\phi},$$

we obtain the following system of first order ODE

$$\frac{dx}{dt} = x(x-y) + n - 2, \qquad \frac{dy}{dt} = x(y-(n-1)x). \tag{4.40}$$

The constant solutions are $(1, n-1)$ and $(-1, -(n-1))$. For g and f to extend smoothly over the origin, we need $\phi(0) = 0$, $\phi'(0) = 1$ and $f'(0) = 0$. Since then $\lim_{r\to 0} t = -\infty$, we have $\lim_{t\to-\infty}(x(t), y(t)) = (1, n-1)$. Linearizing (4.40) at $(1, n-1)$, we have

$$\frac{dX}{dt} = (3-n)X - Y$$
$$\frac{dY}{dt} = -(n-1)X + Y.$$

The eigenvalues of the matrix $\begin{pmatrix} 3-n & -1 \\ -(n-1) & 1 \end{pmatrix}$ are 2 and $2-n$, so we have a saddle point at $(1, n-1)$. Hence there are two solutions with $\lim_{t\to-\infty}(x(t), y(t)) = (1, n-1)$. For one of these solutions we have

$$(x(t), y(t)) \to (0, \infty)$$

as t increases. It can be shown (see Chapter 1 of [**143**]) that this solution defines a complete Ricci soliton g on $\mathbb{R}^n$ of the form (4.37) with positive curvature operator, called the **Bryant soliton** [**57**], [**299**]. Let K_{sph} and

K_{rad} denote the sectional curvatures of planes tangent to the spheres and tangent to the radial direction, respectively. From the formulas for the sectional curvatures (4.38), (4.39), and $C^{-1}r^{1/2} \leq \phi(r) \leq Cr^{1/2}$, $\phi'(r) = O\left(r^{-1/2}\right)$ and $\phi''(r) = O\left(r^{-3/2}\right)$, we have

$$K_{\text{rad}} = O\left(r^{-2}\right), \quad K_{\text{sph}} = O\left(r^{-1}\right).$$

That is, we have the following.

THEOREM 4.14 (Bryant soliton). *For all $n \geq 3$, there exists a unique (up to homothety) complete, steady, gradient Ricci soliton metric on $\mathbb{R}^n$ with positive curvature operator. The eigenspaces of the curvature operator consist of 2-forms that are the wedge product of two 1-forms. The corresponding planes are either tangent to the spheres, in which case the sectional curvatures decay inverse linearly in distance to the origin, or pass through the radial direction, in which case the sectional curvatures decay inverse quadratically.*

The volume of the ball of radius s centered at the origin is

$$\operatorname{Vol}(B(O,s)) = n\omega_n \int_0^s \phi(r)^{n-1} dr.$$

For $s \geq 1$ we have

$$\operatorname{Vol}(B(O,s)) \approx \int_0^s r^{\frac{n-1}{2}} dr \approx s^{\frac{n+1}{2}}.$$

For example, if $n = 3$, then the volume of balls grows quadratically in the radius.

In [**301**], Ivey constructed Ricci solitons on doubly warped products which generalize the Bryant soliton.

The Bryant soliton should appear as a singularity model corresponding to a degenerate neckpinch (see §7.3).

9. Geometry at spatial infinity of ancient solutions

In general, we are interested in the geometry at spatial infinity of limit solutions to the Ricci flow, such as the Bryant soliton (see Chapter 7 for a further discussion of singularities and their limit solutions). To this end, we define the following invariants of the geometry at infinity.

The **asymptotic scalar curvature ratio** (or ASCR) of a complete noncompact Riemannian manifold (M^n, g) is defined by

$$\operatorname{ASCR}(g) = \limsup_{d(x,O)\to\infty} R(x)\, d(x,O)^2,$$

where $O \in M^n$ is any choice of origin. We leave it as an exercise to show that this definition is independent of the choice of O. The **asymptotic**

volume ratio of a complete noncompact Riemannian manifold (M^n, g) with nonnegative Ricci curvature is defined by

$$\mathrm{AVR}\,(g) \doteqdot \lim_{r\to\infty} \frac{\mathrm{Vol}\,(B\,(O,r))}{r^n}.$$

Again, this definition is independent of the choice of O (see Theorem 19.1 of [**267**]).

From the evolution equation for the lengths of paths (6.22), which tells us for any fixed path γ

$$\left| \frac{d}{dt} \log \mathrm{L}_{\,g(t)} \gamma \right| \leq \sup_{M^n} |\mathrm{Rc}\,(g\,(t))|\,,$$

the most obvious bound on how spatial distances change in time is given by the following. If $|\mathrm{Rc}\,(x,t)| \leq \phi\,(t)$, then for any times $t_1 < t_2$ and points x and y

$$e^{-\int_{t_1}^{t_2} \phi(t)dt} d_{t_1}\,(x,y) \leq d_{t_2}\,(x,y) \leq e^{\int_{t_1}^{t_2} \phi(t)dt} d_{t_1}\,(x,y)\,.$$

Hamilton improved this estimate to[1]

$$d_{t_2}\,(x,y) \geq d_{t_1}\,(x,y) - C \int_{t_1}^{t_2} \sqrt{\phi\,(t)}dt. \tag{4.41}$$

Assume all solutions in the following lemmas are complete and noncompact.

THEOREM 4.15 (Asymptotic scalar curvature ratio is independent of time). *If* $(M^n, g\,(t))$ *is an ancient solution with bounded nonnegative curvature operator, then* $\mathrm{ASCR}\,(g\,(t))$ *is independent of* t.

PROOF. See Theorem 19.1 in [**267**]. □

THEOREM 4.16 (Asymptotic scalar curvature ratio is infinite on steady solitons, $n \geq 3$). *Let* (M^n, g_0, f_0) *be a complete steady gradient Ricci soliton:*

$$\mathrm{Rc}\,(g_0) + \mathrm{Hess}_{\,g_0}\,(f_0) = 0$$

such that $\mathrm{sect}\,(g_0) > 0$ *and* R *attains its maximum at some point* $O \in M$ *(which we call the origin). If* $n \geq 3$, *then* $\mathrm{ASCR}\,(g_0) = \infty$.

PROOF. Recall by Theorem 4.1 we can put steady Ricci gradient solitons in canonical form. In particular, there exists a 1-parameter group of diffeomorphisms $\varphi_t : M \to M$ such that

$$g\,(t) = \varphi_t^* g_0$$

is a solution of Ricci flow, for all $t \in (-\infty, \infty)$

$$\mathrm{Rc}\,(g\,(t)) + \mathrm{Hess}_{g(t)}\, f\,(t) = 0$$

where $f\,(t) \doteqdot f \circ \varphi_t$,

$$\frac{\partial}{\partial t} \varphi_t\,(x) = (\mathrm{grad}_{\,g_0} f_0)\,(\varphi_t\,(x))\,,$$

[1] Inequality 4.41 is a trivial generalization of Theorem 17.1 in [**267**].

and

$$\frac{\partial f}{\partial t} = |\nabla f|^2 = \Delta f.$$

We shall prove the theorem by contradiction: *suppose*

$$\operatorname{ASCR}(g(t)) \equiv \operatorname{ASCR}(g_0) < \infty \text{ for all } t.$$

(Note $g(t)$ is isometric to g_0 for all t.) We shall show that there exists a pointwise backward limit

$$g_\infty(x,t) \doteqdot \lim_{t\to-\infty} g(x,t)$$

for all $x \in M - \{O\}$ and that $(M - \{O\}, g_\infty)$ is a complete *flat* Riemannian manifold (although each $g(t)$ is incomplete on $M - \{O\}$.) Since $\sec(g_0) > 0$ implies M^n is diffeomorphic to $\mathbb{R}^n$, $M - \{O\}$ is diffeomorphic to $S^{n-1} \times \mathbb{R}$, we obtain a contradiction when $n \geq 3$.[2]

Recall that since R attains its maximum at O,

$$R + |\nabla f|^2 \equiv R(O).$$

Since $\operatorname{ASCR}(g(t)) < \infty$,

$$\lim_{d_{g(t)}(x,O)\to\infty} R(x,t) = 0. \tag{4.42}$$

Hence

$$\lim_{d_{g(t)}(x,O)\to\infty} |\nabla f|(x,t) = \sqrt{R(O)} > 0.$$

Using $\operatorname{ASCR}(g(t)) < \infty$ again, there exists $C < \infty$ such that for all $x \in M$

$$\begin{aligned} 0 &\leq -\frac{\partial}{\partial t} g(x,t) = 2\operatorname{Rc}(x,t) \\ &\leq 2R(x,t)\, g(x,t) \\ &\leq \frac{C}{1 + d_{g(t)}(x,O)^2} g(x,t) \\ &= \frac{C}{1 + d_{g_0}(\varphi_t(x), O)^2} g(x,t) \end{aligned}$$

($g(t)$ are all isometric to each other by isometries fixing O, so C is independent of t.)

For any $\varepsilon > 0$, there exists $c > 0$ such that

$$d_{g_0}(\varphi_t(x), O) \geq c\,|t| \tag{4.43}$$

for all $x \in M - B_{g_0}(O,\varepsilon)$. Let $\phi_0 = -f_0$ so that

$$\operatorname{Rc}(g_0) = \operatorname{Hess}_{g_0}(\phi_0).$$

[2] Actually $\operatorname{Rc}(g_0) > 0$ implies f is convex, proper, and $M^n \cong \mathbb{R}^n$.

Note that for any $c_0 > \phi_0(O)$ there exists $\delta_0 > 0$ such that if $\phi_0(x) \geq c_0$, then $|\nabla \phi_0|^2(x) \geq \delta_0$. Given any $\varepsilon > 0$, there exists $c_1 > \phi_0(O)$ such that $\{x \in M : f(x) \leq c_1\} \subset B_{g_0}(O, \varepsilon)$. If $x \in M - B_{g_0}(O, \varepsilon)$, then

$$-\frac{\partial}{\partial t}\phi_0(\varphi_t(x)) = |\nabla \phi_0|^2(\varphi_t(x)) \geq 0$$

so that $\varphi_t(x) \in M - B_{g_0}(O, \varepsilon)$ for all $t \leq 0$ and

$$-\frac{\partial}{\partial t}\phi_0(\varphi_t(x)) \geq \delta_0$$

for all $t \leq 0$. From this we conclude (4.43).

The estimate (4.43) implies for any $\varepsilon > 0$ there exists $C < \infty$ such that for all $x \in M - B_{g_0}(O, \varepsilon)$

$$0 \leq -\frac{\partial}{\partial t} g(x,t) \leq \frac{C}{1 + |t|^2} g(x,t).$$

Hence for any $\varepsilon > 0$, there exists $C > 0$ such that

$$Cg(x,0) \geq g(x,t_1) \geq g(x,t_2) \geq g(x,0)$$

for all $x \in M - B_{g_0}(O, \varepsilon)$ and $-\infty < t_1 \leq t_2 \leq 0$. Hence the pointwise limit $\lim_{t \to -\infty} g(x,t) \doteqdot g_\infty$ exists on $M - \{O\}$. Since we have uniform bounds on all derivatives of curvature, g_∞ is smooth. One way to see this is to take a sequence of times $t_i \to -\infty$ and apply Shi's estimates (steady solitons are ancient so the estimates apply to each metric $g(t_i)$ to get derivative of curvature bounds) and the Arzela-Ascoli theorem to get pointwise convergence of a subsequence $g(t_i)$ to g_∞. The Riemannian manifold $(M^n - \{O\}, g_\infty)$ is complete. This is because for any point $x \in M - \{O\}$ and $r < d_{g(t)}(x, O)$, the submanifold $\left(\overline{B_{g(t)}(x,r)}, g(t)\right)$ is complete, and for any $x \in M - \{O\}$,

$$\lim_{t \to -\infty} d_{g(t)}(x, O) = \lim_{t \to -\infty} d_{g_0}(\varphi_t(x), O) = \infty.$$

Furthermore, by (4.42)

$$R_{g_\infty}(x) = \lim_{t \to -\infty} R(x,t) = \lim_{t \to -\infty} R_{g_0}(\varphi_t(x)) = 0$$

for all $x \in M - \{O\}$. Since $\operatorname{sect}(g(t)) > 0$ we have $\operatorname{sect}(g_\infty) \geq 0$ and we conclude $\operatorname{sect}(g_\infty) \equiv 0$ on $M - \{O\}$, from which we obtain our contradiction to the assumption $\operatorname{ASCR}(g_0) < \infty$. □

LEMMA 4.17 (Curvature decay to zero is preserved). *If $(M^n, g(t))$, $t \in [0, T)$, is a solution to Ricci flow with bounded curvature and*

$$\lim_{d_0(x,O) \to \infty} |\operatorname{Rm}(x,0)| = 0,$$

then $\lim_{d_t(x,O) \to \infty} |\operatorname{Rm}(x,t)| = 0$ *all* $t \geq 0$.

PROOF. See Theorem 18.2 in [**267**]. □

PROPOSITION 4.18 (Asymptotic volume ratio is constant). *If* $(M^n, g(t))$ *is a solution with bounded nonnegative Ricci curvature and*

$$\lim_{d(x,O)\to\infty} |\mathrm{Rm}(x,t)| = 0 \text{ for all } t,$$

then $\mathrm{AVR}(g(t))$ *is independent of* t.

PROOF. See Theorem 18.3 in [**267**]. □

For historical reasons, we note the following proposition (see Theorem 19.2 of [**267**].) In Volume 2 we shall see that any ancient solution has $\mathrm{AVR}(g) = 0$. Thus the hypothesis in the following proposition is vacuous. That is, a Type I ancient solution with bounded positive curvature operator has $\mathrm{ASCR}(g) = \infty$.

PROPOSITION 4.19. *If* $(M^n, g(t))$ *is a Type I ancient solution with bounded positive curvature operator and* $\mathrm{ASCR}(g) < \infty$, *then*

(1) (Positive asymptotic volume ratio)

$$\mathrm{AVR}(g) > 0.$$

(2) (At most quadratic decay of curvature) *For any choice of* $O \in M^n$, *there exists* $c = c(O,t) > 0$ *such that*

$$R(x,t)\, d_{g(t)}(x,O)^2 \geq c$$

for all $x \in M^n$.

It is easy to see that for the cigar soliton Σ^2 we have $\mathrm{ASCR}(\Sigma^2) = 0$ and $\mathrm{AVR}(\Sigma^2) = 0$. In particular, the cigar is asymptotic to a cylinder without rescaling (blowing down) and its curvature decays exponentially as a function of distance to the origin.

For the Bryant soliton B^3, we have $\mathrm{ASCR}(B^3) = \infty$ (since the norm of curvature decays inversely proportional to the distance to the origin) and $\mathrm{AVR}(B^3) = 0$. Blow down limits of the Bryant soliton (dimension reduction) are cylinders $S^2 \times \mathbb{R}$.

Blowing down a parabola to a line. A rather trivial but apt analogue of this is the following. Consider the parabola $P = \{(x, x^2) : x \in \mathbb{R}\}$ in the plane. Take a sequence of points $\{x_i\}$ with $x_i \to \infty$. Translate the parabola vertically so that the line $y = x_i^2$ becomes the x-axis and then homothetically rescale the translated parabola so that the distance x_i from $(x_i, 0)$ to the y-axis becomes 1. That is, consider the sequence of parabolas:

$$P_i \doteqdot \left\{ \left(\frac{x}{x_i}, \frac{x^2 - x_i^2}{x_i} \right) : x \in \mathbb{R} \right\}.$$

We can rewrite P_i as

$$P_i = \left\{ \left(\pm\sqrt{1 + \frac{y}{x_i}}, y \right) : y \geq -x_i \right\}.$$

From this we see that P_i limits to the two lines $\{x = \pm 1\} = S^0 \times \mathbb{R}$ as $i \to \infty$ (since $x_i \to \infty$).

10. Homogeneous solutions

We say that a Riemannian manifold (M^n, g) is (globally) **homogeneous** if for every $x, y \in M^n$ there exists an isometry $\iota : M^n \to M^n$ with $\iota(x) = y$. A nice class of homogeneous manifolds are Lie groups with left-invariant metrics. Suppose G is a 3-dimensional **unimodular** (i.e., its volume form is bi-invariant) Lie group with a left invariant metric g. Then there exists a left-invariant frame field $\{f_i\}_{i=1}^3$ with dual coframe field $\{\eta^i\}_{i=1}^3$ such that there are positive constants A, B, C such that metric is diagonal:

$$g = A\eta^1 \otimes \eta^1 + B\eta^2 \otimes \eta^2 + C\eta^3 \otimes \eta^3$$

and the Lie brackets are of the form:

$$[f_i, f_j] = c_{ij}^k f_k \tag{4.44}$$

where $c_{ij}^k \in \{1, 0, -1\}$ and $c_{ij}^k = 0$ unless i, j, k are distinct. Let $\lambda \doteqdot c_{23}^1$, $\mu \doteqdot c_{31}^2$, $\nu \doteqdot c_{12}^3$. The frame field $\{e_i\}_{i=1}^3$ defined by

$$e_1 \doteqdot A^{-1/2} f_1, \quad e_2 \doteqdot B^{-1/2} f_2, \quad e_3 \doteqdot C^{-1/2} f_3,$$

is orthonormal.

Formula (4.44) implies

$$[e_i, e_j] = \frac{\lambda_k c_{ij}^k}{(\lambda_i \lambda_j \lambda_k)^{1/2}} e_k$$

where $\lambda_1 = A$, $\lambda_2 = B$ and $\lambda_3 = C$. By (1.150) the components of the Levi-Civita connection are

$$\begin{aligned}\langle \nabla_{e_i} e_j, e_k \rangle &= \frac{1}{2} \left(\langle [e_i, e_j], e_k \rangle - \langle [e_i, e_k], e_j \rangle - \langle [e_j, e_k], e_i \rangle \right) \\ &= \frac{1}{2(\lambda_i \lambda_j \lambda_k)^{1/2}} \left(\lambda_k c_{ij}^k - \lambda_j c_{ik}^j - \lambda_i c_{jk}^i \right).\end{aligned}$$

Substituting this into (1.152) and since $\nabla_{e_j} e_j = 0$, we have

$$\begin{aligned}\langle \mathrm{Rm}(e_i, e_j) e_j, e_i \rangle &= \left\langle \nabla_{e_i} e_j, \nabla_{e_j} e_i \right\rangle - \left\langle \nabla_{e_j} e_j, \nabla_{e_i} e_i \right\rangle - \left\langle \nabla_{[e_i, e_j]} e_j, e_i \right\rangle \\ &= \frac{1}{4\lambda_i \lambda_j \lambda_k} \left(\lambda_k c_{ij}^k - \lambda_j c_{ik}^j - \lambda_i c_{jk}^i \right) \left(\lambda_k c_{ji}^k - \lambda_i c_{jk}^i - \lambda_j c_{ik}^j \right) \\ &\quad - \frac{1}{2\lambda_i \lambda_j \lambda_k} \lambda_k c_{ij}^k \left(\lambda_i c_{kj}^i - \lambda_j c_{ki}^j - \lambda_k c_{ji}^k \right) \\ &= \frac{1}{4\lambda_i \lambda_j \lambda_k} \left(\left(\lambda_i c_{jk}^i - \lambda_j c_{ki}^j \right)^2 - \left(\lambda_k c_{ij}^k \right)^2 \right) \\ &\quad + \frac{2}{4\lambda_i \lambda_j \lambda_k} \lambda_k c_{ij}^k \left(\lambda_i c_{jk}^i + \lambda_j c_{ki}^j - \lambda_k c_{ij}^k \right)\end{aligned}$$

where $\{i, j, k\}$ is a permutation of $\{1, 2, 3\}$ and we have used the anti-symmetry of c_{ij}^k in i and j. Thus the sectional curvatures $K(e_i \wedge e_j) =$

$\langle \mathrm{Rm}(e_i, e_j)e_j, e_i\rangle$ are given by:

$$K(e_2 \wedge e_3) = \frac{(\mu B - \nu C)^2}{4ABC} + \lambda \frac{2\mu B + 2\nu C - 3\lambda A}{4BC}$$
$$K(e_3 \wedge e_1) = \frac{(\nu C - \lambda A)^2}{4ABC} + \mu \frac{2\nu C + 2\lambda A - 3\mu B}{4AC}$$
$$K(e_1 \wedge e_2) = \frac{(\lambda A - \mu B)^2}{4ABC} + \nu \frac{2\lambda A + 2\mu B - 3\nu C}{4AB}$$

and the $\langle \mathrm{Rm}(e_k, e_i) e_j, e_k\rangle = 0$ for any $i \neq j$ and k. From this we can easily derive that the Ricci tensor is diagonal and given by:

$$\mathrm{Rc}(e_1, e_1) = \frac{(\lambda A)^2 - (\mu B - \nu C)^2}{2ABC} \tag{4.45}$$

$$\mathrm{Rc}(e_2, e_2) = \frac{(\mu B)^2 - (\nu C - \lambda A)^2}{2ABC} \tag{4.46}$$

$$\mathrm{Rc}(e_1, e_1) = \frac{(\nu C)^2 - (\lambda A - \mu B)^2}{2ABC}. \tag{4.47}$$

Hence the Ricci flow equation is equivalent to the following system:

$$\frac{dA}{dt} = \frac{(\mu B - \nu C)^2 - (\lambda A)^2}{BC} \tag{4.48}$$

$$\frac{dB}{dt} = \frac{(\nu C - \lambda A)^2 - (\mu B)^2}{AC} \tag{4.49}$$

$$\frac{dC}{dt} = \frac{(\lambda A - \mu B)^2 - (\nu C)^2}{AB}. \tag{4.50}$$

We note that the normalized Ricci flow (3.4) is:

(4.51)
$$\frac{dA}{dt} = \frac{-4(\lambda A)^2 + 2(\mu B)^2 + 2(\nu C)^2 - 4\mu B \cdot \nu C + 2\nu C \cdot \lambda A + 2\lambda A \cdot \mu B}{3BC}$$
(4.52)
$$\frac{dB}{dt} = \frac{2(\lambda A)^2 - 4(\mu B)^2 + 2(\nu C)^2 + 2\mu B \cdot \nu C - 4\nu C \cdot \lambda A + 2\lambda A \cdot \mu B}{3BC}$$
(4.53)
$$\frac{dC}{dt} = \frac{2(\lambda A)^2 + 2(\mu B)^2 - 4(\nu C)^2 + 2\mu B \cdot \nu C + 2\nu C \cdot \lambda A - 4\lambda A \cdot \mu B}{3BC}$$

since by summing up (4.45)-(4.47) we see that the scalar curvature is

$$R = \frac{-(\lambda A)^2 - (\mu B)^2 - (\nu C)^2 + 2\mu B \cdot \nu C + 2\nu C \cdot \lambda A + 2\lambda A \cdot \mu B}{2ABC}.$$

First we consider the case when the Lie group G is $\mathrm{SU}(2)$ so that there exists a frame such that $\lambda = \mu = \nu = -2$. In this case

$$\begin{aligned} R &= 2\frac{-A^2 - B^2 - C^2 + 2BC + 2CA + 2AB}{ABC} \\ &= 2\frac{A^2 + B^2 + C^2 - (B-C)^2 - (A-C)^2 - (A-B)^2}{ABC}. \end{aligned}$$

Note the special case where $B = C$ in which case $R = 2\frac{2B-A}{B^2}$. In particular, when $B = C < \frac{A}{2}$, we have $R < 0$. Unlike the case of the Ricci flow on surfaces, the scalar curvature does not remain negative on $\mathrm{SU}(2)$. In fact we have the following (see p. 728-9 of [**297**]).

THEOREM 4.20 (Isenberg-Jackson). *For any left-invariant initial metric g_0 on $\mathrm{SU}(2)$ there exists a solution $g(t)$ of the normalized Ricci flow defined for all $t \in [0, \infty)$ with $g(0) = g_0$ such that $g(t)$ converges to a constant positive sectional curvature metric as $t \to \infty$.*

On $\mathrm{SU}(2)$ the normalized Ricci flow equations (4.51)-(4.53) become

$$\begin{aligned} \frac{dA}{dt} &= 4\frac{-4A^2 + 2B^2 + 2C^2 - 4BC + 2CA + 2AB}{3BC} \\ \frac{dB}{dt} &= 4\frac{2A^2 - 4B^2 + 2C^2 + 2BC - 4CA + 2AB}{3BC} \\ \frac{dC}{dt} &= 4\frac{2A^2 + 2B^2 - 4C^2 + 2BC + 2CA - 4AB}{3BC}. \end{aligned}$$

Under the normalized flow $(ABC)(t)$ is independent of time. Hence we may assume without loss of generality $(ABC)(t) \equiv 8/3$. We then have

$$\frac{dA}{dt} = A\left(A(B + C - 2A) + (B - C)^2\right) \tag{4.54}$$

$$\frac{dB}{dt} = B\left(B(A + C - 2B) + (A - C)^2\right) \tag{4.55}$$

$$\frac{dC}{dt} = C\left(C(A + B - 2C) + (A - B)^2\right). \tag{4.56}$$

From this we may compute the evolution equations for the difference of the metric components:

$$\begin{aligned} \frac{d}{dt}(A - C) &= 2C^3 - 2A^3 + AB^2 + A^2B - BC^2 - B^2C \\ &= \left(-2\left(A^2 + AC + C^2\right) + B^2 + B(A + C)\right)(A - C) \end{aligned} \tag{4.57}$$

and similarly for $\frac{d}{dt}(A - B)$ and $\frac{d}{dt}(B - C)$. We assume without loss of generality

$$A(0) \geq B(0) \geq C(0).$$

By the above equations, we have

$$A(t) \geq B(t) \geq C(t)$$

for all $t > 0$. Since $A + B - 2C \geq 0$, equation (4.56) implies $\frac{dC}{dt} \geq 0$ so that $C(t) \geq C(0)$.

Now we may estimate the factor on the RHS of (4.57)

$$\begin{aligned}
& -2\left(A^2 + AC + C^2\right) + B^2 + B(A + C) \\
& = -2C^2 - \left(A^2 - B^2\right) - AC - (A + C)(A - B) \\
& \leq -2C^2 \leq -2C(0)^2 .
\end{aligned}$$

Thus

$$\frac{d}{dt}(A - C) \leq -2C(0)^2 (A - C).$$

We conclude that

$$A(t) - C(t) \leq (A(0) - C(0))\, e^{-2C(0)^2 t}$$

for all $t > 0$. That is, $A - C$ decays exponentially to zero. Since $A \geq B \geq C$ and we have normalized the volume so that $ABC \equiv 8/3$, we conclude that $A(t), B(t), C(t)$ exponentially converge to $A_\infty = B_\infty = C_\infty \doteqdot 2/\sqrt[3]{3}$. That is, $g(t)$ exponentially converges as $t \to \infty$ to the constant sectional curvature metric

$$g_\infty \doteqdot \left(2/\sqrt[3]{3}\right)\left(\eta^1 \otimes \eta^1 + \eta^2 \otimes \eta^2 + \eta^3 \otimes \eta^3\right).$$

Second we consider the case where G is the Heisenberg group (Nil) of upper-triangular 3×3 real matrices. In this case there is a frame where $\lambda = -2$ and $\mu = \nu = 0$. Then for Nil the Ricci flow equations (4.48)-(4.50) are equivalent to:

$$-\frac{d}{dt}\log A = \frac{d}{dt}\log B = \frac{d}{dt}\log C = 4\frac{A}{BC}. \tag{4.58}$$

We immediately see that A is decreasing whereas B and C are increasing. In fact B/C, AB and AC are independent of time. We compute

$$\frac{d}{dt}\log\left(\frac{A}{BC}\right) = -12\frac{A}{BC}$$

so that

$$\frac{A}{BC}(t) = \frac{1}{12}\left(\frac{B_0 C_0}{12 A_0} + t\right)^{-1}$$

where $A_0 \doteqdot A(0)$, $B_0 \doteqdot B(0)$, $C_0 \doteqdot C(0)$. Thus one can explicitly solve (4.58) to get

$$\frac{A_0}{A(t)} = \frac{B(t)}{B_0} = \frac{C(t)}{C_0} = \left(1 + \frac{12A_0}{B_0 C_0}t\right)^{1/3} \leq \text{const} \cdot (t+1)^{1/3}. \tag{4.59}$$

The sectional curvatures:

$$K(e_2 \wedge e_3) = -\frac{3A}{BC} = -3\left(\frac{B_0C_0}{A_0} + 12t\right)^{-1} \tag{4.60a}$$

$$K(e_3 \wedge e_1) = \frac{A}{BC} = \left(\frac{B_0C_0}{A_0} + 12t\right)^{-1} \tag{4.60b}$$

$$K(e_1 \wedge e_2) = \frac{A}{BC} = \left(\frac{B_0C_0}{A_0} + 12t\right)^{-1} \tag{4.60c}$$

satisfy $|\mathrm{sect}(g(t))| \leq \mathrm{const} \cdot t^{-1}$. Note that the scalar curvature is negative as it must be for the solution to exist for all time. If we consider a compact quotient of the Heisenberg group, such as $G/\mathbb{Z}^3$, the diameters satisfy $\mathrm{diam}(g(t)) \leq \mathrm{const} \cdot (t+1)^{1/6}$. Hence

$$|\mathrm{sect}(g(t))| \, \mathrm{diam}(g(t))^2 \leq \mathrm{const} \cdot t^{-2/3} \to 0 \quad \text{as } t \to \infty. \tag{4.61}$$

Thus the solution **collapses** as time tends to infinity. In fact, (4.61) says that the metrics become more and more **almost flat**. The only nonzero bracket is $[e_2, e_3] = -\sqrt{\frac{A}{BC}}e_1 \approx t^{-1/2}e_1$, so that the brackets tend to zero when measured in an orthonormal frame.

See [**86**], [**297**], [**274**], [**320**] and [**324**] for further discussion of the Ricci flow of homogeneous 3-manifolds and related **quasi-stability** questions (some in the category of warped products). Besides $\mathrm{SU}(2)$ and Nil, some other 3-dimensional homogeneous geometries correspond to Solv, $\widetilde{\mathrm{SL}}(2,\mathbb{R})$, $\widetilde{\mathrm{Isom}}(\mathbb{R}^2)$ and $\mathbb{E}^3$. In the case of Solv and $\widetilde{\mathrm{SL}}(2,\mathbb{R})$, homogeneous solutions exist for all time and $|\mathrm{Rm}| \leq Ct^{-1}$. In the case of $\widetilde{\mathrm{Isom}}(\mathbb{R}^2)$ the solution exists for all time and $|\mathrm{Rm}| \leq Ce^{-ct}$ while for $\mathbb{E}^3$ the solution is flat and hence stationary. These geometries roughly correspond to Thurston's model geometries, but are not in 1-1 correspondence, since the geometries he considers have maximal isotropy groups. See Isenberg-Jackson-Lu [**298**] for work on the Ricci flow on 4-dimensional homogeneous spaces.

EXERCISE 4.21. *Another way to compute the curvatures of a left-invariant metric on a Lie group is as follows. Suppose that G is a Lie group and $\{f_i\}_{i=1}^n$ is a left-invariant frame field with*

$$[f_i, f_j] = c_{ij}^k f_k$$

and $g = a_i^2 \eta^i \otimes \eta^i$, where $\{\eta^i\}_{i=1}^n$ is the dual coframe to $\{f_i\}_{i=1}^n$ (i.e., $\eta^i(f_j) = \delta_j^i$.) We compute

$$\begin{aligned} d\eta^i(f_j, f_k) &= f_j\left(\eta^i(f_k)\right) - f_k\left(\eta^i(f_j)\right) - \eta^i([f_j, f_k]) \\ &= -c_{jk}^i. \end{aligned}$$

Thus

$$d\eta^i = -c_{jk}^i \eta^j \wedge \eta^k.$$

An orthonormal frame field is $\{e_i\}$ *where* $e_i = a_i^{-1} f_i$ *and the dual coframe is* $\{\omega^i\}$ *where* $\omega^i = a_i \eta^i$*. Since*

$$d\omega^i (e_j, e_k) = -\frac{a_i}{a_j a_k} c_{jk}^i,$$

we have by (1.52)

$$\begin{aligned} \omega_i^k &= \frac{1}{2}\Big(d\omega^i (e_j, e_k) + d\omega^j (e_i, e_k) - d\omega^k (e_j, e_i)\Big)\,\omega^j \\ &= \frac{1}{2}\left(-\frac{a_i}{a_j a_k} c_{jk}^i - \frac{a_j}{a_i a_k} c_{ik}^j + \frac{a_k}{a_i a_j} c_{ji}^k\right)\omega^j. \end{aligned}$$

The we may use the second Cartan structure equation to compute the curvature 2-form

$$\Omega_i^k = d\omega_i^k - \omega_i^j \wedge \omega_j^k.$$

Compute $\langle \mathrm{Rm}(e_i, e_j)\, e_k, e_\ell \rangle$.

11. Isometry group

If $(M^n, g(t))$ is a solution to the Ricci flow on a closed manifold and γ is an isometry of M^n with respect to the metric $g(0)$, then γ is also an isometry of M^n with respect to $g(t)$ for all $t > 0$. This follows from the uniqueness of solutions of the Ricci flow with a given initial metric. For if γ is an isometry of $g(0)$, then $g(t)$ and $\gamma^* g(t)$ are both solutions of the Ricci flow with initial metric $g(0)$. Hence by uniqueness, $g(t) = \gamma^* g(t)$ for all $t > 0$.

Problem 4.22 (A. Fisher). *If* $(M^n, g(t))$ *is a solution to the Ricci flow on a closed manifold, then is* $\mathrm{Isom}(g(t)) = \mathrm{Isom}(g(0))$ *for all* $t > 0$*?*

By the above remarks, we see that $\mathrm{Isom}(g(t)) \supset \mathrm{Isom}(g(0))$. Since we find it hard to believe that a spontaneous symmetry can be created, we conjecture that we actually have equality. Note the problem easily reduces to a backward uniqueness question:

Problem 4.23. *If* $g_1(t)$ *and* $g_2(t)$ *are solutions to the Ricci flow on a closed manifold on a time interval* $[0, T)$ *and if* $g_1(t') = g_2(t')$ *for some* $t' \in (0, T)$, *then is* $g_1(t) = g_2(t)$ *for all* $t \in [0, t']$ *(and hence for all* $t \in [0, T)$*)?*

Indeed, if γ is an isometry of $g(t')$, then $g(t)$ and $\gamma^* g(t)$ are solutions of the Ricci flow with $g(t') = \gamma^* g(t')$; an affirmative answer to the latter problem would imply $g(t) = \gamma^* g(t)$ for all t. In particular, γ would be an isometry of $g(0)$.

12. Notes and commentary

§5. The analogue to the cigar soliton for the **curve shortening flow** (or **CSF** for short) of a plane curve $\frac{\partial x}{\partial t} = -\kappa \nu$, where κ is the curvature and ν is the unit outward normal, is the **grim reaper** translating (steady) soliton:

$$y = t + \log \sec x \tag{4.62}$$

for $x \in (-\pi/2, \pi/2)$ and $t \in \mathbb{R}$. The CSF was first proposed by Mullins [**393**] to model the motion of idealized grain boundaries and he also discovered the grim reaper solution (a terminology later coined by Matt Grayson). For curves which are graphs of functions $y = f(x,t)$, the curve shortening flow becomes:

$$\frac{\partial y}{\partial t} = \left(1 + \left(\frac{\partial y}{\partial x}\right)^2\right)^{-1} \frac{\partial^2 y}{\partial x^2}.$$

From this one easily checks that the curves given by (4.62) are solutions to the CSF. If we let θ denote the angle the unit tangent vector makes with the x-axis, then we have

$$\kappa = \cos\theta.$$

Next we consider steady soliton solutions of the one-dimensional heat equation of the form $u(x,t) = F\left(\frac{x}{\sqrt{t}}\right)$. The heat equation leads to the ODE

$$F''(y) = -\frac{1}{2} y F'(y)$$

where $y = \frac{x}{\sqrt{t}}$. Thus, for some $A \in \mathbb{R}$

$$F'(y) = Ae^{-\frac{1}{4}y^2}.$$

We conclude

$$u(x,t) = B + A\sqrt{\pi}\operatorname{erf}\left(\frac{x}{2\sqrt{t}}\right)$$

for some $A, B \in \mathbb{R}$ and where

$$\operatorname{erf}(y) \doteqdot \frac{2}{\sqrt{\pi}} \int_0^y e^{-w^2} dw.$$

Note that

$$u(x, t+s) = u\left(\frac{\sqrt{t}}{\sqrt{t+s}} x, t\right).$$

§6. The analogue of the Rosenau solution for the CSF is

$$y = \log\left(\cos x + \sqrt{e^{2t} + \cos^2 x}\right) - t \tag{4.63}$$

or $-\infty < t < 0$. This solution can be derived as follows. The evolution of the curvature $\kappa(\theta,t)$ is given by $\kappa_t = \kappa^2 \kappa_{\theta\theta} + \kappa^3$. Searching for a solution of the form $\kappa(\theta,t)^2 = a(\theta) + b(t)$ leads to the equations

$$a''(\theta) + 4a(\theta) = 0, \quad \left(a'(\theta)\right)^2 + 4a(\theta)^2 = 4C^2, \quad b'(t) - 2b(t)^2 = -2C^2$$

where C is a constant. Taking $C = 1$, we have the particular solution $a(\theta) = \cos 2\theta$ and $b(t) = \coth(-2t)$ for $t < 0$. Hence the solution satisfies

$$\kappa(\theta,t) = \sqrt{\cos 2\theta + \coth(-2t)}.$$

From this we can derive (4.63). Note that

$$\lim_{t\to-\infty}\kappa(\theta,t)=\sqrt{2}\left|\cos\theta\right|.$$

This exhibits the fact that the limit as $t\to-\infty$ at either end of the oval is the grim reaper (without rescaling).

In Olwell [**409**] translating solutions to the Gauss curvature flow are constructed.

§7. Recall that graph solutions $y=y(x,t)$ of the curve shortening flow satisfy

$$y_t=\frac{y_{xx}}{y_x^2+1}. \tag{4.64}$$

We look for solutions of the form

$$y(x,t)=\sqrt{t}F\left(\frac{x}{\sqrt{t}}\right). \tag{4.65}$$

Such solutions

$$y\left(x,c^2t\right)=cy\left(\frac{x}{c},t\right).$$

We compute

$$y_t=-\frac{x}{2t}F'\left(\frac{x}{\sqrt{t}}\right)+\frac{1}{2\sqrt{t}}F\left(\frac{x}{\sqrt{t}}\right),$$
$$y_x=F'\left(\frac{x}{\sqrt{t}}\right)$$

and

$$y_{xx}=\frac{1}{\sqrt{t}}F''\left(\frac{x}{\sqrt{t}}\right)$$

Under the assumption (4.65), setting $r\doteqdot\frac{x}{\sqrt{t}}$, we find that (4.64) is equivalent to

$$-\frac{r}{2}F'(r)+\frac{1}{2}F(r)=\frac{F''(r)}{F'(r)^2+1}.$$

We may rewrite this as the second order ODE

$$2F''(r)+r\left[F'(r)\right]^3-\left[F'(r)\right]^2F(r)+rF'(r)-F(r)=0. \tag{4.66}$$

LEMMA 4.24. *There exists a solution of (4.66) with boundary values:*

$$\lim_{r\to0}F(r)=\infty\quad\textit{and}\quad\lim_{r\to\infty}F(r)=0.$$

This solution is called the 90° ***Brakke wedge****.*

EXERCISE 4.25. *Prove the above lemma.*

CHAPTER 5

Analytic results and techniques

In this chapter we discuss various analytic results and techniques that are fundamental for the study of the qualitative behavior of solutions of the Ricci flow. This includes derivative estimates – useful for proving long time existence of solutions and obtaining some local control of solutions, Cheeger-Gromov type compactness theorem for solutions of Ricci flow – necessary for obtaining pointed limits of dilations about singularities, Hamilton-Ivey estimate – used for showing singularity models formed in finite time in dimension 3 have nonnegative sectional curvature, strong maximum principle for systems – used to obtain splitting and rigidity of these singularity models. In the last two sections we give some applications of the strong maximum principle for systems to the study of solutions with nonnegative curvature.

1. Derivative estimates and long time existence

The following estimate says that the **maximum of the curvatures cannot grow too fast too soon** as time increases.

LEMMA 5.1 (Doubling time estimate). *If* $|\mathrm{Rm}(g(0))| \leq K$, *then*

$$|\mathrm{Rm}(g(t))| \leq 2K$$

for all $0 \leq t \leq 1/(16K)$.

PROOF. From (3.6) we see that

$$\begin{aligned} \frac{\partial}{\partial t}|\mathrm{Rm}|^2 &= \Delta|\mathrm{Rm}|^2 - 2|\nabla \mathrm{Rm}|^2 + 8(B_{ijk\ell} + B_{ikj\ell})R_{ijk\ell} \\ &\leq \Delta|\mathrm{Rm}|^2 - 2|\nabla \mathrm{Rm}|^2 + 16|\mathrm{Rm}|^3, \end{aligned} \tag{5.1}$$

where $B_{ijk\ell} \doteqdot -R_{pij}^{q}R_{q\ell k}^{p}$. Let $\rho(t)$ be the solution to the corresponding ODE

$$\frac{d}{dt}\rho^2 = 16\rho^3, \quad \rho(0) = K.$$

By the maximum principle (Lemma 2.11), we have

$$|\mathrm{Rm}(g(t))| \leq \rho(t) = \frac{K}{1 - 8Kt}.$$

□

So far we have seen the nice cases of Ricci flow where we have global existence of the normalized flow and convergence to a constant sectional curvature metric. Now we shall consider the more prevalent case, which is when singularities occur.

DEFINITION 5.2 (Singular solution). *Given a solution* $(M^n, g(t))$, $t \in [0,T)$, *of the Ricci flow we say that* $[0,T)$, *is the* ***maximal time interval*** *of existence if for every solution* $(M^n, \tilde{g}(t))$, $t \in [0, \tilde{T})$, *of the Ricci flow with* $\tilde{g}(t) = g(t)$ *for* $t \in [0, \min\left\{T, \tilde{T}\right\})$ *we have* $\tilde{T} \leq T$. *Moreover, if* $T < \infty$, *then we say that* $g(t)$ ***forms a singularity*** *at time* T.

The following result says that the solution exists as long as the curvature remains bounded.

PROPOSITION 5.3 (Long time existence). *If* $(M^n, g(t))$ *is a solution of the Ricci flow on a closed manifold on a maximal time interval* $[0,T)$, *where* $T < \infty$, *then*

$$\sup_{M^n \times [0,T)} |\mathrm{Rm}| = \infty. \tag{5.2}$$

In fact,

$$\boxed{\lim_{t \to T} \max_{M^n} |\mathrm{Rm}(\cdot, t)| = \infty.} \tag{5.3}$$

REMARK 5.4. *Lemma 5.1 may be used to show that (5.2) implies (5.3). The contrapositive statement is obtained as follows. Suppose there exists a sequence of times* $t_i \to T$ *and a constant* $K < \infty$ *such that*

$$\max_{M^n} |\mathrm{Rm}(\cdot, t_i)| \leq K$$

for all $i \in \mathbb{N}$. *Then, by the doubling time estimate, we have* $|\mathrm{Rm}(g(t))| \leq 2K$ *for all* $t_i \leq t \leq t_i + 1/(16K)$. *From this we conclude that* $\sup_{M^n \times [t_j, T)} |\mathrm{Rm}| \leq 2K$ *for some* $j \in \mathbb{N}$.

The idea of the proof is to obtain **higher derivative of curvature estimates**. Solutions with a given bound on the curvature may have covariant derivatives that are initially arbitrarily large. However, for short time, one can obtain bounds for the higher covariant derivatives of the curvature which improve with time on a short enough initial time interval (see [**153**], p. 223-4).

EXAMPLE 5.5 (Bumpy metrics). *Consider the 1-parameter family of initial metrics on* $\mathbb{R} \times S^1$

$$g_\varepsilon \doteqdot dr^2 + \left(1 + \varepsilon \sin\left(r/\sqrt{\varepsilon}\right)\right)^2 d\theta^2.$$

We have

$$K(g_\varepsilon) = \frac{\sin(r/\sqrt{\varepsilon})}{1 + \varepsilon \sin(r/\sqrt{\varepsilon})} = 1 - \frac{1}{1 + \varepsilon \sin(r/\sqrt{\varepsilon})}$$

which is uniformly bounded for $\varepsilon \in (0, 1/2]$. *Note that*

$$|\nabla K(g_\varepsilon)| = \left|\frac{\partial}{\partial r} K(g_\varepsilon)\right| = \frac{1}{\sqrt{\varepsilon}} \frac{|\cos(r/\sqrt{\varepsilon})|}{(1 + \varepsilon \sin(r/\sqrt{\varepsilon}))^2}$$

so that

$$\sup_{\mathbb{R}\times S^1} |\nabla K(g_\varepsilon)| \geq \frac{1}{\sqrt{\varepsilon}}$$

which tends to infinity as $\varepsilon \to 0$. *Since* sine *is a periodic function, we can quotient these metrics to yields metrics on* $S^1 \times S^1$. *It is nice to think of the following theorem applying to these initial metrics.*

THEOREM 5.6 (Global derivative of curvature estimates). *If* $(M^n, g(t))$, $t \in [0,T)$, *is a solution of the Ricci flow on a closed manifold, then for each* $\alpha > 0$ *and every* $m \in \mathbb{N}$ *, there exists a constant* $C(m,n,\alpha)$ *depending only on* m*, and* n*, and* $\max\{\alpha, 1\}$*such that if*

$$|\mathrm{Rm}(x,t)|_{g(t)} \leq K \qquad \text{for all } x \in M^n \text{ and } t \in [0, \frac{\alpha}{K}] \cap [0,T),$$

then

(5.4)

$$|\nabla^m \mathrm{Rm}(x,t)|_{g(t)} \leq \frac{C(m,n,\alpha)K}{t^{m/2}} \qquad \text{for all } x \in M^n \text{ and } t \in (0, \frac{\alpha}{K}] \cap [0,T).$$

We call these estimates **Bernstein-Bando-Shi estimates** (see [**38**], [**462**] and [**463**]; for related smoothing estimates, see [**42**]). For $m = 1$, we can prove (5.4) by considering the gradient quantity

$$F = t|\nabla \mathrm{Rm}|^2 + \beta |\mathrm{Rm}|^2,$$

and showing

$$\frac{\partial F}{\partial t} \leq \Delta F + C(n)\beta K^3, \tag{5.5}$$

for β depending only on $\max\{\alpha, 1\}$ and n, and where the constant $C(n)$ depends only on n. Since $F \leq \beta K^2$ at $t = 0$, by applying the maximum principle to (5.5), we obtain $t|\nabla \mathrm{Rm}|^2 \leq F \leq C(n,\alpha)K^2$ for $t \in [0, \alpha/K]$, where $C(n,\alpha)$ depends only on $\max\{\alpha, 1\}$ and n. The first derivative estimate immediately follows. The higher derivative estimates can be proved using analogous quantities involving higher derivatives of curvature. We refer the reader to [**153**], Theorem 7.1 on p. 223-4.

EXERCISE 5.7. *Fill in the details of the proof of (5.5) for* $m = 1$. HINT*: show that (see (5.1))*

$$\frac{\partial}{\partial t}|\mathrm{Rm}|^2 \leq \Delta |\mathrm{Rm}|^2 - 2|\nabla \mathrm{Rm}|^2 + 16|\mathrm{Rm}|^3$$

and

$$\frac{\partial}{\partial t}|\nabla \mathrm{Rm}|^2 \leq \Delta |\nabla \mathrm{Rm}|^2 + C|\mathrm{Rm}| \cdot |\nabla \mathrm{Rm}|^2$$

where we have dropped the good $-2\left|\nabla^2 \mathrm{Rm}\right|^2$ *term from the* RHS.

We now give the statement of the following estimate of W.-X. Shi.

THEOREM 5.8 (Local derivative of curvature estimates). *For any* α, K, r, n *and* $m \in \mathbb{N}$, *there exists* C *depending only on* α, K, r, n *and* m *such that if* M^n *is a manifold,* $p \in M$, *and* $g(t)$, $t \in [0, T_0]$, $0 < T_0 \leq \alpha/K$, *is a solution to the Ricci flow on an open neighborhood* U *of* p *containing* $\bar{B}_{g(0)}(p, r)$ *as a compact subset, and if*

$$|\mathrm{Rm}(x,t)| \leq K \text{ for all } x \in U \text{ and } t \in [0, T_0],$$

then

$$|\nabla^m \mathrm{Rm}(y,t)| \leq \frac{C(\alpha, K, r, n, m)}{t^{m/2}}$$

for all $y \in B_{g(0)}(p, r/2)$ *and* $t \in (0, T_0]$.

The proof of Proposition 5.3 is based on this and the following elementary result which gives a sufficient condition for the metrics to remain uniformly equivalent to the initial metric (see Lemma 14.2 of [**255**] and [**153**], p. 203).

LEMMA 5.9 (Uniform equivalence of the metrics). *Let* $g(t)$, $t \in [0, T)$, *where* $T \leq \infty$, *be a smooth 1-parameter family of metrics on a manifold* M^n. *If there exists a constant* $C < \infty$ *such that*

$$\int_0^T \sup_{x \in M^n} \left| \frac{\partial g}{\partial t}(x,t) \right|_{g(t)} dt \leq C, \tag{5.6}$$

then for any $x_0 \in M^n$ *and* $t_0 \in [0, T)$, *we have*

$$e^{-C} g(x_0, 0) \leq g(x_0, t_0) \leq e^{C} g(x_0, 0). \tag{5.7}$$

Moreover, the metrics $g(t)$ *converge uniformly as* $t \to T$ *to a continuous metric* $g(T)$ *with* $e^{-C} g \leq g(T) \leq e^{C}$.

Applying the first part of this to a solution to the Ricci flow, we have

COROLLARY 5.10 (Bounded Ricci implies uniform equivalence). *If*

$$\sup_{M \times [0,T)} |\mathrm{Rc}| \leq K,$$

then

$$e^{-2KT} g(x, 0) \leq g(x, t) \leq e^{2KT} g(x, 0),$$

for all $x \in M^n$ *and* $t \in [0, T)$.

PROOF OF THE LEMMA. Given $V \in T_x M$ and times $0 \leq t_1 \leq t_2 < T$, we have

(5.8)

$$\left| \log \frac{g_{(x,t_2)}(V,V)}{g_{(x,t_1)}(V,V)} \right| = \left| \int_{t_1}^{t_2} \frac{\frac{\partial}{\partial t} g_{(x,t)}(V,V)}{g_{(x,t)}(V,V)} dt \right| \leq \int_{t_1}^{T} \left| \frac{\partial g}{\partial t}(x,t) \right|_{g(t)} dt \doteqdot C(t_1) \leq C,$$

and $\lim_{t \to T} C(t) = 0$. Taking the exponential of this estimate with $t_1 = 0$ yields (5.7). By (5.8), the continuity of the metrics $g(t)$, and the formula

$$g(t)(V, W) = \frac{1}{4} \left(|V + W|_{g(t)}^2 - |V - W|_{g(t)}^2 \right),$$

we see that for every $V, W \in TM$

$$\lim_{t \to T} g(t)(V, W) \doteqdot g(T)(V, W)$$

exists; the convergence is uniform on compact sets, and $g(T)(V, W)$ is a metric continuous in V and W. □

EXERCISE 5.11. *Prove the following. If* $\mathrm{Rc} \geq -K$, *then* $g(x,t) \leq e^{2KT} g(x,0)$. *If* $\mathrm{Rc} \leq K$, *then* $g(x,t) \geq e^{-2KT} g(x,0)$.

In Volume 2 we present Sesum's theorem [**453**] which states that a solution to the Ricci flow on a closed manifold exists as long as the Ricci curvature is bounded. That is, if $|\mathrm{Rc}| \leq C$ on a time interval $[0, T)$, where $T < \infty$, then the solution can be continued past time T.

To obtain convergence of the metric, we need to bound its derivatives. This is the reason for stating the following (see also Proposition 6.48 on p. 203 of [**153**].)

PROPOSITION 5.12 (Curvature bound implies metric derivative bounds). *If* $(M^n, g(t))$ *is a solution of the Ricci flow on a closed manifold with*

$$|\mathrm{Rm}(x,t)|_{g(t)} \leq K$$

for all $x \in M^n$, $t \in [0, T)$, *and some* $K < \infty$, *then for any background metric* $\bar{g}$ *and* $m \in \mathbb{N}$, *there exists a constant* $C_m < \infty$ *depending only on* m, K, T, $g(0)$ *and* $\bar{g}$ *such that*

$$\left|\bar{\nabla}^m g(x,t)\right|_{\bar{g}} \leq C_m \qquad \text{for all } x \in M^n \text{ and } t \in [0, T), \tag{5.9}$$

where $\bar{\nabla}$ *is the Riemannian covariant derivative associated to* $\bar{g}$.

This completes the sketch of the proof of Proposition 5.3.

REMARK 5.13. *Note that we have not assumed that the derivatives of the curvature are bounded. By Shi's estimate such uniform estimates hold on any time interval* $[\varepsilon, T)$, *where* $\varepsilon > 0$.

2. Cheeger-Gromov type compactness theorem for Ricci flow

In the study of singular solutions, we are interested in taking limits of sequences of solutions of the Ricci flow. Such sequences are obtained by dilating about a sequence of points and times. Before we consider this case of time-dependent metrics, we first consider sequences of pointed Riemannian manifolds. Let $\{(M_i^n, g_i, O_i)\}_{i \in \mathbb{N}}$ be a sequence of complete Riemannian manifolds with a uniform bound on the curvatures over the whole manifolds and a lower bound on the injectivity radius at the origins:

(1) $|\mathrm{Rm}(g_i)| \leq C$ on M_i^n
(2) $\mathrm{inj}(g_i, O_i) \geq c > 0$.

Since we are interested in smooth convergence, we also assume bounds on the higher covariant derivatives of curvature:

3. $\left|\nabla^k \operatorname{Rm}(g_i)\right| \leq C_k$ on M_i^n, where ∇^k denotes the kth covariant derivative with respect to g_i.

The **Cheeger-Gromov compactness theorem** says the following.

THEOREM 5.14 (Cheeger-Gromov compactness for pointed Riemannian manifolds). *There exists a subsequence which* ***converges in*** C^∞ *to a complete Riemannian manifold* $(M_\infty^n, g_\infty, O_\infty)$ *with* $|\operatorname{Rm}(g_\infty)| \leq C$ *on* M_∞^n, $\operatorname{inj}(g_\infty, O_\infty) \geq c > 0$, *and* $\left|\nabla^k \operatorname{Rm}(g_\infty)\right| \leq C_k$ *on* M_∞^n.

By **convergence in** C^∞ we mean that there exists an exhaustion $\{U_i\}_{i\in\mathbb{N}}$ of M_∞^n by open sets and diffeomorphisms $F_i : U_i \to M_i^n$ onto an open subset of M_i^n such that $F_i(O_\infty) = O_i$ and the sequence of metrics $\tilde{g}_i \doteqdot F_i^*(g_i)$ converges to g_∞ on each open set K with compact closure in each C^k-norm[1] (measured with respect to g_∞.) Note that for each K, we have $K \subset U_i$ for i large enough.

REMARK 5.15. *Instead of saying for a sequence there exists a subsequence which converges we shall sometimes say that a sequence* ***preconverges****.*

EXAMPLE 5.16 (2d - capping cylinders). *Consider the sequence* (M_i^2, g_i) *of Riemannian surfaces diffeomorphic to* S^2 *given as follows. Take the finite cylinder* $S^1 \times [-i, i]$, *where* S^1 *is the unit circle, and attach two unit hemispheres to each end. This metric is not* C^∞ *so smooth the metrics out in a way independent of i and invariant under reflection about the center circle* $S^1 \times \{0\}$.

(1) *If* $O_i = (x_i, y_i)$ *is a sequence of points such that* $\min\{i - y_i, y_i + i\} \leq C$ *independent of* i, *then after passing to a subsequence the pointed sequence* (M_i^2, g_i, O_i) *converges to* M_∞^2 *equal to* $S^1 \times [0, \infty)$ *with the unit hemisphere attached at* $S^1 \times \{0\}$ *and the metric smoothed out. In this case* M_∞^2 *is diffeomorphic to* $\mathbb{R}^2$. *For convenience, let's now assume* $y_i + i \leq C$. *The subsequence is chosen so that* $\lim_{i\to\infty}(y_i + i)$ *exists. We may take the exhaustion* U_i *of* M_∞^2 *to be* $S^1 \times [0, i)$ *with the unit hemisphere attached at* $S^1 \times \{0\}$ *and the metric smoothed out. The diffeomorphism* $F_i : (U_i, g_\infty) \to (M_i^2, g_i)$ *is an isometry into half of* M_i^2.
(2) *On the other hand if* $O_i = (x_i, y_i)$ *is such that*
$$\lim_{i\to\infty} \min\{i - y_i, y_i + i\} = \infty,$$
then (M_i^2, g_i, O_i) *converges to* $M_\infty^2 = S^1 \times \mathbb{R}$, *which is a cylinder. We leave it to the reader to fill in the details.*

The **compactness theorem for the Ricci flow** [**268**] states that given a sequence of complete solutions to the Ricci flow with bounded curvature

[1]For sequences of solutions of the Ricci flow we give a more technical description of this below.

and an injectivity radius estimate at the origins, there exists a subsequence which converges to a complete solution. There is a long history in this subject and we refer the reader to [**422**] for a much more detailed discussion. We now give the statement of the result we need. The main applications are to the study of singularities where sequences of solutions arise from dilating about a sequence of points and times approaching the singularity. Typically the understanding (classification) of the resulting limits are derived from monotonicity formulas. In fact, Perelman's no local collapsing Theorem provides the injectivity radius estimate necessary to obtain a noncollapsed limit.

Let $p \in \mathbb{N}$ and K be a compact subset in a Riemannian manifold (M^n, g). We say that a sequence of tensors $T_i(t)$, $t \in [0,T]$, defined on K **converges** to $T_\infty(t)$ in the $C^p(K \times [0,T], g)$-topology as $i \to \infty$ if in every normal coordinate system $\left(x^1, \ldots, x^n\right)$ in a ball B centered at a point $x \in K$, we have

$$\frac{\partial^{\alpha+\ell} T_i(t)}{\partial x^\alpha \partial t^\ell} \to \frac{\partial^{\alpha+\ell} T_\infty(t)}{\partial x^\alpha \partial t^\ell}$$

uniformly on $(B \cap K) \times [0,T]$ for $|\alpha| + 2\ell \leq p$. Here $\alpha = (\alpha_1, \ldots, \alpha_k)$ is a multi-index and $\partial x^\alpha \doteqdot \partial x^{\alpha_1} \cdots \partial x^{\alpha_k}$ and $|\alpha| \doteqdot \alpha_1 + \cdots + \alpha_k$.

DEFINITION 5.17 (Convergence of manifolds). *Let $p \in \mathbb{N}$ and $\{M_i^n, g_i(t), O_i\}$, $t \in [0,T]$, with $T > 0$ and $O_i \in M_i^n$ be a sequence of evolving pointed Riemannian manifolds. We say $\{M_i^n, g_i(t), O_i\}$, $t \in [0,T]$,* ***converges to an evolving pointed Riemannian manifold*** *$\{M_\infty, g_\infty(t), O_\infty\}$, $t \in [0,T]$, in the $C^p(g_\infty(0))$-topology if there exists a sequence of open sets U_i in M_∞ containing O_∞ satisfying $U_i \subset U_{i+1}$ and $\cup_i U_i = M_\infty$, and a sequence of diffeomorphisms F_i mapping the sets U_i to open sets V_i in M_i^n satisfying (i) $F_i(O_\infty) = O_i$, and (ii) for any compact set K in M_∞ the pulled back metrics $\tilde{g}_i(t) \doteqdot F_i^* g_i(t)$ converge to $g_\infty(t)$ in the $C^p(K \times [0,T], g_\infty(0))$-topology.*

First we state a local version of the compactness theorem for the Ricci flow.

THEOREM 5.18 (Compactness theorem - local). *Let $\{(M_i^n, g_i(t), O_i)\}_{i \in \mathbb{N}}$, $t \in [0,T]$, with $T > 0$ be a sequence of complete pointed solutions to the Ricci flow. Let $p_0 \geq 4$ be an integer and $s_0 > 0$. Suppose that we have:*

(i) *the uniform (derivative of) curvature bounds*

$$\sup_{B_{g_i(0)}(O_i, s_0) \times [0,T]} |\nabla^q \operatorname{Rm}(g_i(t))| \leq C_{q,s_0} < \infty$$

for all $0 \leq q \leq p_0$, and

(ii) *an injectivity radius bound*

$$\operatorname{inj}_{g_i(0)}(O_i) \geq \delta > 0$$

for all $i \in \mathbb{N}$.

Then there exists $c(n) < \infty$ *and a subsequence of*

$$\left\{ \left(B_{g_i(0)} \left(O_i, e^{-c(n)TC_{0,s_0}} s_0 \right), g_i(t), O_i \right) \right\}_{i \in \mathbb{N}}, \quad t \in [0,T],$$

which converges to an evolving pointed Riemannian manifold $\{B_\infty^n, g_\infty(t), O_\infty\}$, $t \in [0,T]$, *in the* $C^{p_0-2}(g_\infty(0))$*-topology and* $g_\infty(t)$ *is a solution of the Ricci flow.*

Furthermore, if we assume the global bounds

$$\sup_{M_i^n \times [0,T]} |\nabla^q \operatorname{Rm}(g_i(t))| \leq C_q < \infty$$

for all $0 \leq q \leq p_0$, *instead of (i), then there exists a subsequence of* $\{(M_i^n, g_i(t), O_i)\}_{i \in \mathbb{N}}$, $t \in [0,T]$, *which converges to an evolving pointed complete Riemannian manifold* $\{M_\infty^n, g_\infty(t), O_\infty\}$, $t \in [0,T]$ *in the* $C^{p_0-2}(g_\infty(0))$*-topology and* $g_\infty(t)$ *is a solution of the Ricci flow.*

For more discussion see [**267**] or [**143**]. Note that when applying the above theorem to the Ricci flow, the bounds in (i) can often be derived from Shi's local derivative of curvature estimates.

The following global compactness theorem is more often used in the study of the Ricci flow.

THEOREM 5.19 (Compactness theorem - global). *Let* $\{(M_i^n, g_i(t), O_i)\}_{i \in \mathbb{N}}$, $t \in (\alpha, \omega) \ni 0$, *be a sequence of complete pointed solutions to the Ricci flow such that*

(1) (uniformly bounded curvatures)

$$|\operatorname{Rm}(g_i(t))|_{g_i(t)} \leq C \quad \text{on } M_i^n \times (\alpha, \omega)$$

for some constant $C < \infty$ *independent of* i, *and*

(2) (injectivity radius estimate at $t = 0$)

$$\operatorname{inj}_{g_i(0)}(O_i) \geq \delta > 0.$$

Then there exists a subsequence of $\{(M_i^n, g_i(t), O_i)\}_{i \in \mathbb{N}}$ *which converges as* $i \to \infty$ *to a pointed complete solution to the Ricci flow* $(M_\infty^n, g_\infty(t), O_\infty)$, $t \in (\alpha, \omega)$, *with* $|\operatorname{Rm}(g_\infty)|_{g_\infty} \leq C$ *on* $M_\infty^n \times (\alpha, \omega)$.

For the proof see [**268**] or [**143**]. Note that this theorem only supposes bounds on the curvature and not on the derivatives of curvature. This is because for the Ricci flow, if the curvature is bounded in space-time, then all the derivatives of the curvature are bounded away from time $t = \alpha$. We do not need the derivative bounds to be independent of time since we may apply a diagonalization argument to obtain a subsequence which converges on the whole time interval (α, ω). For this same reason, we may weaken the hypothesis of the theorem and assume only that the curvatures are bounded by a continuous function of time: $|\operatorname{Rm}(g_i(t))|_{g_i(t)} \leq C(t)$.

REMARK 5.20. *The argument could be extended with some care to solutions restricted to geodesic balls or tubes. The only major difficulty is that the geodesic balls and tubes at time* $t = 0$ *do not stay geodesic balls or tubes* *for $t > 0$. However one can estimate how fast the geodesics balls or tubes are changing and thus one can find geodesic balls or tubes (of a smaller radius) at a later time which are contained in the balls or tubes at time $t = 0$. For simplicity, we omit the statements and proofs of these extensions (see* [**143**]*).*

Some related works on compactness theorems for the Ricci flow are Lu [**356**] on the compactness theorem for orbifolds, and Glickenstein [**229**] on the compactness theorem without an injectivity radius assumption. See also Carfora and Marzuoli [**87**].

EXERCISE 5.21 (Backward limit of Rosenau as cigar or cylinder). *Consider the Rosenau ancient solution of Chapter 4, §6. Determine for which sequences of points and times (x_i, t_i) with $t_i \to -\infty$ the time translated solutions $\left(S^2, g_i(t), x_i\right)$, where $g_i(t) \doteqdot g(t_i + t)$, converges to:*

(1) *the cigar soliton solution,*
(2) *the flat cylinder solution.*

2.1. Shorter proof of the classification of closed 3-manifolds with positive Ricci curvature. Using the compactness theorem we can shorten Hamilton's proof that closed 3-manifolds M^3 with positive Ricci curvature are diffeomorphic to spherical space forms. However we do not prove here, as Hamilton does, that the solution of the Ricci flow converges *exponentially fast in C^∞* to a constant positive sectional curvature metric. Instead we prove the convergence after rescaling of a *sequence* of metrics $g(t_i)$ to a constant positive sectional curvature metric for some $t_i \to T$. This is enough to deduce that M^3 is a spherical space form. The basic estimates that the proof relies on are as follows. For a solution to the Ricci flow on a closed 3-manifold with positive Ricci curvature initially:

(1) Nonnegative Ricci and positive scalar curvature are preserved ((3.27) and Corollary 2.10).
(2) The 'pinching improves' estimate (5.10).
(3) The no local collapsing theorem and the consequent local injectivity radius estimate (see Volume 2).
(4) The strong maximum principle applied to solutions of the Ricci flow with nonnegative scalar curvature everywhere and positive scalar curvature at a point; this is applied on a limit solution (Corollary 5.37).
(5) Shi's local derivative estimates (Theorem 5.8).
(6) The Cheeger-Gromov compactness theorem for sequences of pointed Riemannian manifolds (C^∞ version: Theorem 5.14).
(7) The contracted second Bianchi identity (1.17).

THEOREM 5.22. *If $\left(M^3, g_0\right)$ is a closed 3-manifold with positive Ricci curvature, then M^3 is diffeomorphic to a spherical space form.*

PROOF. By the point picking methods in subsections 2.2.1 and 2.2.2 of 7.2.2.2, the local injectivity radius estimate corollary of the no local collapsing theorem, the derivative of curvature estimates, and the compactness theorem, there exists a sequence of points and times (x_i, t_i) with $t_i \to T$ such that

$$K_i \doteqdot |\mathrm{Rm}(x_i, t_i)| = \max_{M^3} |\mathrm{Rm}(t_i)|$$

and the pointed rescaled solutions $(M^3, g_i(t), x_i)$, where

$$g_i(t) \doteqdot K_i g\left(t_i + K_i^{-1} t\right),$$

converge in C^∞ on compact sets to a complete solution $\left(M^3, g_\infty(t), x_\infty\right)$, $t \in (-\infty, \omega)$, $\omega > 0$, with bounded nonnegative Ricci curvature (by the Hamilton-Ivey estimate below we actually know that the limit solution has nonnegative sectional curvature, but we do not need this) and $|\mathrm{Rm}(x_\infty, 0)| = 1$. (At the moment we do not know whether M^3_∞ is compact or not, this will be proved later.) This implies $R(g_\infty(x_\infty, 0)) > 0$. Since $R(g_\infty(t)) \geq 0$, by the strong maximum principle, we have $R(g_\infty(t)) > 0$ on all of M_∞. Fix any $\rho > 0$ so that we have uniform positive lower bound $R(g_\infty(0)) \geq c > 0$ in $B_{g_\infty(0)}(x_\infty, \rho)$. We shall show that $\mathrm{Rc} \equiv \frac{1}{3} R g$ for the metric $g_\infty(0)$ in $B_{g_\infty(0)}(x_\infty, \rho)$. Recall by the 'Ricci pinching improves' estimate, that there exists $\delta > 0$ and $C < \infty$ such that for the original solution $g(t)$

$$\frac{\left|\mathrm{Rc} - \frac{1}{3} R g\right|}{R} \leq C R^{-\delta}. \tag{5.10}$$

Now $g_i(0) = K_i g(t_i)$ converges to $g_\infty(0)$ in C^∞ on compact sets. Hence for i large enough, we have

$$R(g_i)(x, 0) \geq \frac{1}{2} \inf_{B_{g_\infty(0)}(x_\infty, \rho)} R(g_\infty(0)) \geq \frac{c}{2}$$

for $x \in B_{g_i(0)}(x_i, \rho - 1)$. This implies

$$R(g(t_i)) = K_i R(g_i(0)) \geq \frac{c}{2} K_i \text{ in } B_{g_i(0)}(x_i, \rho - 1) = B_{g(t_i)}\left(x_i, \frac{\rho - 1}{\sqrt{K_i}}\right).$$

Now by (5.10), the scale-invariant measure of the difference from Einstein satisfies:

$$\begin{aligned} \frac{\left|\mathrm{Rc} - \frac{1}{3} R g\right|}{R}(g_i(0)) &= \frac{\left|\mathrm{Rc} - \frac{1}{3} R g\right|}{R}(g(t_i)) \\ &\leq C R(g(t_i))^{-\delta} \leq C\left(\frac{c}{2} K_i\right)^{-\delta} \end{aligned}$$

in $B_{g_i(0)}(x_i, \rho - 1)$. Again using the convergence of $g_i(0)$ to $g_\infty(0)$, we find that

$$\frac{\left|\mathrm{Rc} - \frac{1}{3} R g\right|}{R}(g_\infty(0)) \leq 2C\left(\frac{c}{2} K_i\right)^{-\delta}$$

in $B_{g_\infty(0)}(x_\infty, \rho - 2)$ for all i sufficiently large. Since $\lim_{i \to \infty} K_i = \infty$, we have $\left|\mathrm{Rc} - \frac{1}{3} R g\right|(g_\infty(0)) \equiv 0$ in $B_{g_\infty(0)}(x_\infty, \rho - 2)$. Since $\rho > 0$ is arbitrary, we conclude that $\mathrm{Rc} \equiv \frac{1}{3} R g$ on M_∞^3 for the metric $g_\infty(0)$. Recall by the contracted second Bianchi identity, this implies $R(g_\infty(0)) \equiv \text{const} > 0$. Hence (for example Myers' Theorem can used) M_∞^3 is compact and $\left(M_\infty^3, g_\infty(0)\right)$ is a spherical space form (in dimension 3 the Weyl tensor is zero so that Einstein metrics have constant sectional curvature). Finally, since M_∞^3 is compact and admits a complete metric which is a limit of a sequence of metrics on M^3, we conclude M_∞^3 is diffeomorphic to M^3. Therefore we have proved the existence of a C^∞ metric with constant positive sectional curvature on M^3. □

3. The Hamilton-Ivey curvature estimate

We now recall a curvature estimate of Hamilton and Ivey, which implies that in a scaled sense, the sectional curvatures of any solution to the Ricci flow on a closed 3-manifold tends to nonnegative. The motivation for the form of this estimate comes from considering the case where $\left(M^3, g(t)\right)$ is rotationally symmetric and forming a neck pinch. In the region of the neck, the two smallest eigenvalues are equal and negative and the largest eigenvalue is positive: $\lambda_1(\mathrm{Rm}) = \lambda_2(\mathrm{Rm}) < 0 < \lambda_3(\mathrm{Rm})$. These inequalities are preserved under the following ODE system

$$\frac{d\lambda_3}{dt} = \lambda_3^2 + \lambda_1^2$$
$$\frac{d\lambda_1}{dt} = \lambda_1^2 + \lambda_1\lambda_3,$$

which we obtain from (3.23) and where $\lambda_i = \lambda_i(\mathbf{M})$. Solving the homogeneous system

$$\frac{d\lambda_1}{d\lambda_3} = \frac{\lambda_1^2 + \lambda_1\lambda_3}{\lambda_3^2 + \lambda_1^2},$$

we have

$$\log(-\lambda_1) = \frac{\lambda_3}{-\lambda_1} + 2\log\left(\frac{-\lambda_1}{\lambda_3 - \lambda_1}\right) + C \tag{5.11}$$

for some constant C. From this we can deduce that if $-\lambda_1$ is sufficiently large, then

$$\lambda_3 > -\lambda_1 \log(-\lambda_1). \tag{5.12}$$

EXERCISE 5.23. *Confirm equation (5.11) and deduce (5.12).*

Before we state the **Hamilton-Ivey estimate**, we recall an analogue of Theorem 3.17, which holds for K depending on t. We adopt the same notation as before.

PROPOSITION 5.24 (Maximum principle for time-dependent convex sets). *Let $g(t)$, $t \in [0, T)$, be a solution to the Ricci flow on a closed manifold M^n and let $K(t) \subset E$ be subsets which are invariant under parallel translation* *and whose intersections* $K(t)_x \doteqdot K(t) \cap E_x$ *with each fiber are closed and convex. Suppose also that the set*

$$\{(v,t) \in E \times [0,T) : v \in K(t)\}$$

is closed in $E \times [0,T)$ *and suppose the* ODE *(3.22) has the property that for any* $\mathbf{M}(t_0) \in K(t_0)$, *we have* $\mathbf{M}(t) \in K(t)$ *for all* $t \in [t_0, T)$. *If* $\mathrm{Rm}(0) \in K$, *then* $\mathrm{Rm}(t) \in K$ *for all* $t \in [0,T)$.

We refer to [**155**] or [**143**] for the proof.

THEOREM 5.25 (Hamilton-Ivey 3-d curvature estimate). *Let* $\left(M^3, g(t)\right)$ *be a solution of the Ricci flow on a closed* 3*-manifold for* $0 \leq t < T$. *If* $\lambda_1(\mathrm{Rm})(x,0) \geq -1$ *for all* $x \in M^3$, *then at any point* $(x,t) \in M^3 \times [0,T)$ *where* $\lambda_1(\mathrm{Rm})(x,t) < 0$, *we have*

$$\boxed{R \geq |\lambda_1(\mathrm{Rm})| \left(\log|\lambda_1(\mathrm{Rm})| + \log(1+t) - 3\right).} \tag{5.13}$$

In particular,

$$R \geq |\lambda_1(\mathrm{Rm})| \left(\log|\lambda_1(\mathrm{Rm})| - 3\right). \tag{5.14}$$

REMARK 5.26. *Assuming* $\lambda_1(\mathrm{Rm}) \geq -1$ *at* $t = 0$, *if* $\lambda_1(\mathrm{Rm})(x,t) < -e^3$, *then* $R(x,t) > 0$.

REMARK 5.27. *By scaling the solution, we see that if we assume*

$$\lambda_1(\mathrm{Rm})(x,0) \geq -C$$

for all $x \in M^3$, *where* $C > 0$, *then*

$$R \geq |\lambda_1(\mathrm{Rm})| \left(\log|\lambda_1(\mathrm{Rm})| + \log\left(C^{-1} + t\right) - 3\right) \tag{5.15}$$

at all (x,t) *where* $\lambda_1(\mathrm{Rm})(x,t) < 0$.

PROOF. Define

$$K(t) \doteqdot \left\{ \mathbf{M} : \begin{array}{c} \lambda_1 + \lambda_2 + \lambda_3 \geq -3/(1+t) \\ \text{and if } \lambda_1 \leq -1/(1+t)\text{, then} \\ \lambda_1 + \lambda_2 + \lambda_3 \geq -\lambda_1 \left(\log(-\lambda_1) + \log(1+t) - 3\right) \end{array} \right\} \subset E.$$

For each t, $K(t)$ is invariant under parallel translation and for each (x,t), $K_x(t)$ is closed and convex (see [**271**] or [**153**], p. 258-260). By our assumption, it is easy to see that $\mathrm{Rm}[g(0)] \in K(0)$. The theorem will follow from showing $\mathrm{Rm}[g(t)] \in K(t)$ for all $t \in [0,T)$. Indeed, if $-1/(1+t) < \lambda_1(\mathrm{Rm}) < 0$, then (5.13) follows directly, and when $\lambda_1(\mathrm{Rm}) \leq -1/(1+t)$, then (5.13) follows from $\mathrm{Rm}[g(t)] \in K(t)$.

From dropping the factor of 2 on the RHS of (3.24), we have

$$\frac{d}{dt}(\lambda_1 + \lambda_2 + \lambda_3) \geq \frac{1}{3}(\lambda_1 + \lambda_2 + \lambda_3)^2.$$

This implies

$$\lambda_1 + \lambda_2 + \lambda_3 \geq -3/(1+t) \tag{5.16}$$

is preserved under the ODE (3.23); that is, if (5.16) holds at time t_0, then (5.16) holds for $t \geq t_0$.

Given $\mathbf{M}$ with $\lambda_1 < 0$, define

$$\phi(\mathbf{M}) = \frac{\lambda_1 + \lambda_2 + \lambda_3}{-\lambda_1} - \log(-\lambda_1).$$

One can compute that under the ODE (3.23)

$$\frac{d\phi}{dt} \geq -\lambda_1. \tag{5.17}$$

We verify this: by (3.23), we calculate

$$\lambda_1^2 \frac{d\phi}{dt} = -\lambda_1^3 - \lambda_1 \left(\lambda_2^2 + \lambda_3^2 + \lambda_2\lambda_3\right) + (\lambda_2 + \lambda_3)\lambda_2\lambda_3.$$

If $\lambda_2 < 0$, then

$$\lambda_1^2 \frac{d\phi}{dt} = -\lambda_1^3 - \lambda_2^3 + (\lambda_2 - \lambda_1)\left(\lambda_2^2 + \lambda_3^2 + \lambda_2\lambda_3\right) \geq -\lambda_1^3.$$

On the other hand, if $\lambda_2 \geq 0$, then

$$\lambda_1^2 \frac{d\phi}{dt} = -\lambda_1^3 - \lambda_1\lambda_2^2 + (\lambda_2 - \lambda_1)\left(\lambda_3^2 + \lambda_2\lambda_3\right) \geq -\lambda_1^3.$$

By (5.17), if $\lambda_1(\mathbf{M}) \leq -1/(1+t)$, then

$$\frac{d}{dt}\left(\frac{\lambda_1 + \lambda_2 + \lambda_3}{-\lambda_1} - \log(-\lambda_1) - \log(1+t)\right) \geq 0.$$

Clearly, this is the estimate we need to show that $K(t)$ is preserved by the ODE (3.23). By Proposition 5.24, we conclude $\mathrm{Rm}[g(t)] \in K(t)$ for all $t \in [0, T)$. $\square$

From (5.14) we see that if $\lambda_1(\mathrm{Rm}) \leq -e^{C+3}$ for some constant $C > 0$, then

$$|\lambda_1(\mathrm{Rm})| \leq C^{-1} R \leq 3C^{-1}\lambda_3(\mathrm{Rm}).$$

That is, **if we have a large negative sectional curvature** (suppose C is large), **then we have a much larger** (in magnitude) **positive sectional curvature**. Note also that if $\lambda_1(\mathrm{Rm}) \leq -e^6$, then

$$R \geq \frac{1}{2}|\lambda_1(\mathrm{Rm})| \log|\lambda_1(\mathrm{Rm})|.$$

Since the function

$$\psi(x) \doteqdot x(\log x - 3)$$

which occurs in (5.14) has derivatives $\psi'(x) = \log x - 2$ and $\psi''(x) = 1/x$. We have $\psi : \left(e^2, \infty\right) \to \left(-e^2, \infty\right)$ is strictly increasing, strictly convex and $\lim_{x\to\infty} \psi(x) = \infty$. From (5.14) we also see that if $\lambda_1(\mathrm{Rm}) < -e^2$, then

$$\lambda_1(\mathrm{Rm}) = -|\lambda_1(\mathrm{Rm})| \geq -\psi^{-1}(R) = -\phi(R) R,$$

where $\phi : (0, \infty) \to (0, \infty)$ is defined by $\phi(u) = \frac{\psi^{-1}(u)}{u}$. Note that

$$\lim_{u\to\infty} \phi(u) = \lim_{x\to\infty} \frac{x}{\psi(x)} = 0.$$

Since $\frac{x}{\psi(x)} = \frac{1}{\log x - 3}$ is decreasing for $x > e^3$, we have $\phi(u)$ is decreasing for $u > 0$. In summary we have:

COROLLARY 5.28 (3d solutions are ϕ-solutions). *There exists a decreasing function $\phi : (0,\infty) \to (0,\infty)$ with $\lim_{u\to\infty} \phi(u) = 0$ such that if $\left(M^3, g(t)\right)$, $t \in [0,T)$, is a solution of the Ricci flow on a closed 3-manifold with $\lambda_1(\mathrm{Rm})(x,0) \geq -1$ for all $x \in M^3$, then*

$$\lambda_1(\mathrm{Rm}) \geq -\phi(R)\, R - e^2 \tag{5.18}$$

on $M^3 \times [0,T)$.

Note that without any normalizing assumptions on the initial metric g_0 at time 0, by (5.15) we see that if $t \geq \varepsilon > 0$ (i.e., the solution has existed for at least a little while), then

$$R \geq |\lambda_1(\mathrm{Rm})| \left(\log |\lambda_1(\mathrm{Rm})| + \log \varepsilon - 3\right) \tag{5.19}$$

wherever $\lambda_1(\mathrm{Rm}) < 0$. Thus on all of $M^3 \times [\varepsilon, T)$ we have

$$\lambda_1(\mathrm{Rm}) \geq -\phi(R)\, R - \varepsilon^{-1} e^2.$$

3.1. 2d and 3d ancient solutions have nonnegative sectional curvature. From (5.19) we obtain the following.

LEMMA 5.29 (3-d ancient solutions have nonnegative sectional curvature). *If we have a complete ancient solution on a 3-manifold with bounded curvatures, then the sectional curvature is nonnegative.*

PROOF. For otherwise, if there exists a point and time where $\lambda_1(\mathrm{Rm}) < 0$, then we have (5.19) for any $\varepsilon > 0$. Taking the limit as $\varepsilon \to \infty$, we obtain a contradiction. □

Considering 3-manifolds which are the product of a surface and a line or a circle, we obtain the following.

COROLLARY 5.30 (2-d ancient solutions have nonnegative curvature). *Complete ancient solutions on surfaces with bounded curvature have nonnegative curvature.*

Actually, it is much simpler to see this from the evolution equation $\frac{\partial}{\partial t} R = \Delta R + R^2$ which holds when $n = 2$. For a very similar reason, in all dimensions ancient solutions with bounded curvature have nonnegative scalar curvature (see Lemma 7.4).

REMARK 5.31. *Going back to Theorem 5.25, we see that if we have a solution defined for all $t \in [0,\infty)$, then if we have a negative sectional curvature $\lambda_1(\mathrm{Rm})$ such that $t\,|\lambda_1(\mathrm{Rm})|$ is large, then there exists a positive sectional curvature $\lambda_3(\mathrm{Rm})$ such that $\lambda_3(\mathrm{Rm}) / |\lambda_1(\mathrm{Rm})|$ is large. However, the homogeneous solutions on* Nil, *which are defined for $t \in [0,\infty)$, give examples of Type III singular solutions on closed 3-manifolds ($t\,|\mathrm{Rm}| \leq C$) where $\lambda_3(\mathrm{Rm}) / |\lambda_1(\mathrm{Rm})|$ is uniformly bounded.*

3.2. Type I and II 3-dimensional dilations have nonnegative sectional curvature. Let $(M^n, g(t))$, $t \in [0,T)$, be a **singular solution** on a closed manifold. That is, either 1. $T = \infty$ or 2. $T < \infty$ and $\sup_{M^n \times [0,T)} |\mathrm{Rm}| = \infty$. (Here we think of the $T = \infty$ case as singular for the purpose of studying singularity models; see Chapter 7.) Suppose that (x_i, t_i), $x_i \in M^n$, $t_i \to T$, is a sequence of points and times and α_i, β_i, and C are positive constants with $\limsup_{i\to\infty} \alpha_i > 0$, $\limsup_{i\to\infty} \beta_i > 0$ and such that

$$|\mathrm{Rm}| \leq CK_i$$

in $B_{g(t_i)}\left(x_i, \alpha_i/\sqrt{K_i}\right) \times [t_i - \beta_i/K_i, t_i]$, where $K_i \doteqdot |\mathrm{Rm}(x_i, t_i)|$. If $T < \infty$, then there exists a subsequence such that $\lim_{i\to\infty} \alpha_i = \alpha_\infty \in (0, \infty]$, $\lim_{i\to\infty} \beta_i = \beta_\infty \in (0, \infty]$, and the dilated solutions

$$\left(B_{g_i(0)}(x_i, \alpha_i), g_i(t), x_i\right), \ t \in (-\beta_i, 0],$$

where $g_i(t) \doteqdot K_i g\left(t_i + K_i^{-1} t\right)$, converge in C^∞ on compact sets to a solution $(B_\infty^n, g_\infty(t), x_\infty)$, $t \in (-\beta_\infty, 0]$, of the Ricci flow with bounded curvature and such that $\overline{B_{g_\infty(0)}(x_\infty, r)} \subset B_\infty^n$ is compact for every $r < \alpha_\infty$. This follows from Perelman's no local collapsing theorem, Shi's local derivative estimates, and Hamilton's Cheeger-Gromov compactness theorem (Theorem 5.18) for Ricci flow. Note that if $\alpha_\infty = \infty$, then $(B_\infty^n, g_\infty(0))$ is complete, and hence $(B_\infty^n, g_\infty(t))$ is complete for all $t \in (-\beta_\infty, 0]$.

A consequence of the Hamilton-Ivey estimate is the following.

LEMMA 5.32 (Finite time singularity models have sect ≥ 0). *If $\left(M^3, g(t)\right)$, $t \in [0,T)$, is a singular solution on a closed 3-manifold with $T < \infty$ and if $\{(x_i, t_i)\}$ is a sequence satisfying the conditions above and such that $|\mathrm{Rm}(x_i, t_i)| \to \infty$, then the limit solution $\left(B_\infty^3, g_\infty(t), x_\infty\right)$, $t \in (-\beta_\infty, 0]$, has nonnegative sectional curvature.*

PROOF. We leave the proof as an exercise for the reader. □

In the case of solutions which exist for infinite time we have:

LEMMA 5.33 (Type IIb singularity models have sect ≥ 0). *If $\left(M^3, g(t)\right)$, $t \in [0,\infty)$, is a solution to the Ricci flow on a closed 3-manifold and if $\{(x_i, t_i)\}$, $t_i \to \infty$, is a sequence as above and such that*

$$\lim_{i\to\infty} t_i |\mathrm{Rm}(x_i, t_i)| = \infty,$$

then the limit solution $\left(B_\infty^3, g_\infty(t), x_\infty\right)$, $t \in (-\beta_\infty, 0]$, has nonnegative sectional curvature if it exists (we need an injectivity radius estimate to get existence of the limit).

PROOF. See [**146**]. □

4. Strong maximum principles and splitting theorems

4.1. The time derivative of the sup function. In proving various versions the maximum principle for systems, it is useful to consider the time derivative of the supremum function. In this short section we present this tool. This section is not necessary for later parts of the book and may be skipped.

Let Y be a **sequentially compact** topological space (i.e., for every sequence of points $\{y_i\}$ in Y, there exists a subsequence which converges). Suppose that a function

$$g : (a,b) \times Y \to \mathbb{R}$$

and its partial time-derivative $\frac{\partial g}{\partial t}(t,y)$ are continuous. Define $h : (a,b) \to \mathbb{R}$ by

$$h(t) \doteqdot \sup_{y \in Y} g(t,y),$$

and

$$\frac{d^+ h}{dt}(t) \doteqdot \limsup_{s \to 0_+} \frac{h(t+s) - h(t)}{s}.$$

LEMMA 5.34 (Maximum principle tool - $\frac{d}{dt}$ of the sup function). *Let $Y_t \doteqdot \{z \in Y : h(t) = g(t,z)\}$. We have for any $t \in (a,b)$*

$$\frac{d^+ h}{dt}(t) = \sup_{y \in Y_t} \frac{\partial g}{\partial t}(t,y).$$

REMARK 5.35. *In applications we will usually only need $\leq$.*

PROOF. Given any sequence $s_i \to 0_+$, let $y_i \in Y$ be such that $h(t+s_i) = g(t+s_i, y_i)$. For any subsequence, since Y is sequentially compact, there exists a further subsequence i_j such that $y_{i_j} \to y_\infty \in Y$ as $j \to \infty$. By the continuity of g on $(a,b) \times Y$, we have

$$g\left(t + s_{i_j}, y_{i_j}\right) \to g(t, y_\infty).$$

We claim $y_\infty \in Y_t$. If $y_\infty \notin Y_t$, then choose a point $y' \in Y_t$ and let

$$g\left(t, y'\right) - g(t, y_\infty) \doteqdot 3\varepsilon > 0.$$

By the continuity of g there exists N_0 such that for $j \geq N_0$ we have

$$\left| g\left(t + s_{i_j}, y_{i_j}\right) - g(t, y_\infty) \right| \leq \varepsilon$$

and there exists $\delta > 0$ such that for $|s| \leq \delta$ we have

$$\left| g\left(t, y'\right) - g\left(t+s, y'\right) \right| \leq \varepsilon.$$

Choose $j \geq N_1$ large enough so that $\left|s_{i_j}\right| \leq \delta$ for $j \geq N_1$. If $j \geq \max\{N_0, N_1\}$, then

$$g\left(t, y'\right) \leq g\left(t + s_{i_j}, y_{i_j}\right) + \varepsilon \leq g(t, y_\infty) + 2\varepsilon = g\left(t, y'\right) - \varepsilon,$$

which is a contradiction. Hence $y_\infty \in Y_t$ so that

$$h(t + s_i) = g(t + s_i, y_i) \to g(t, y_\infty) = h(t)$$

and we have proved that $h(t)$ is a continuous function.

Now since $g(t, y_{i_j}) \leq g(t, y_\infty)$,

$$\begin{aligned}\limsup_{j\to\infty} \frac{h(t+s_{i_j}) - h(t)}{s_{i_j}} &= \limsup_{j\to\infty} \frac{g(t+s_{i_j}, y_{i_j}) - g(t, y_\infty)}{s_{i_j}} \\ &\leq \limsup_{j\to\infty} \frac{g(t+s_{i_j}, y_{i_j}) - g(t, y_{i_j})}{s_{i_j}} \\ &= \frac{\partial g}{\partial t}(t, y_\infty) \leq \sup_{y\in Y_t} \frac{\partial g}{\partial t}(t,y).\end{aligned}$$

The last equality follows from the continuity of $\frac{\partial g}{\partial t}$ and the Mean Value Theorem

$$\frac{g(t+s_{i_j}, y_{i_j}) - g(t, y_{i_j})}{s_{i_j}} = \frac{\partial g}{\partial t}\left(t + s'_{i_j}, y_{i_j}\right)$$

for some $0 \leq s'_{i_j} \leq s_{i_j}$. Hence for any $s_i \to 0_+$

$$\limsup_{i\to\infty} \frac{h(t+s_i) - h(t)}{s_i} \leq \sup_{y\in Y_t} \frac{\partial g}{\partial t}(t,y).$$

The lemma will follow from proving the reverse inequality. Since Y_t is a closed subset of a sequentially compact space, there exists $y' \in Y_t$ such that $\frac{\partial g}{\partial t}(t, y') = \sup_{y\in Y_t} \frac{\partial g}{\partial t}(t,y)$. Then for any $s_i \to 0_+$

$$\limsup_{i\to\infty} \frac{h(t+s_i) - h(t)}{s_i} \geq \limsup_{i\to\infty} \frac{g(t+s_i, y') - g(t, y')}{s_i} = \frac{\partial g}{\partial t}(t, y').$$

□

4.2. The strong maximum principle and local splitting.

4.2.1. *Strong maximum principle for scalar heat-type equations.* First recall the strong maximum principle for scalar solutions. It says that if we are given a nonnegative solution of a heat-type equation, and if the solution (temperature) is positive at some point and time, then the temperature is positive at all points and times except possibly at the initial time. See Protter-Weinberger [**431**] or Landis [**335**].

THEOREM 5.36. *Suppose that $u : M^n \times [0,T) \to \mathbb{R}$ is a classical solution to the following heat-type equation with respect to a continuous 1-parameter family of (not necessarily complete) C^1 metrics $g(t)$*

$$\frac{\partial u}{\partial t} \geq \Delta_{g(t)} u + X(t) \cdot \nabla u + c(t)\, u$$

where $X(x,t)$ is a continuous time-dependent vector field and $c(x,t)$ is a continuous time-dependent function. If $u \geq 0$ on $M \times [0,T)$ and $u(x_0, t_0) > 0$ for some $x_0 \in M$ and $t_0 > 0$, then $u(x,t) > 0$ for all $x \in M$ and $t \in (t_0 - \varepsilon, T)$ for some $\varepsilon > 0$.

Since under the Ricci flow, $\frac{\partial R}{\partial t} = \Delta R + 2|\mathrm{Rc}|^2 \geq \Delta R$, we have the following.

COROLLARY 5.37. *Suppose that* $(M^n, g(t))$, $t \in [0,T)$, *is a (not necessarily complete) solution to the Ricci flow with* $R(g(t)) \geq 0$ *for all* $t \geq 0$ *and* $R(x_0, t_0) > 0$ *for some* $x_0 \in M$ *and* $t_0 > 0$. *Then* $R(x,t) > 0$ *for all* $x \in M$ *and* $t \in (t_0 - \varepsilon, T)$ *for some* $\varepsilon > 0$.

REMARK 5.38. *Note that the arguments in the proof of the strong maximum principle are local and only rely on the (path) connectedness of the manifold* M *and not on the completeness of the metric* g.

EXERCISE 5.39. *Show that if* M^n *is closed, then we can improve Corollary 5.37 to obtain the following conclusion:* $R(x,t) > 0$ *for all* $x \in M$ *and* $t \in (0,T)$.

SOLUTION. Suppose $R(x_1, t_1) = 0$ for some $x_1 \in M$ and $t_1 \in (0, t_0)$. Then we have $R(x,t) \equiv 0$ for all $x \in M$ and $t \in (0, t_1)$. In particular $\mathrm{Rc} \equiv 0$ on $M \times (0, t_1)$. By the uniqueness of solutions of the Ricci flow on closed manifolds, we have $\mathrm{Rc} \equiv 0$ on $M \times (0,T)$. This contradicts $R(x_0, t_0) > 0$.

4.2.2. *Strong maximum principle for* Rm *under Ricci flow.* Recall that the Cheeger-Gromoll splitting Theorem 1.77 says that if (M^n, g) is a complete Riemannian manifold with nonnegative Ricci curvature and admits a geodesic line, then (M^n, g) splits as the Riemannian product of a line and an $(n-1)$-manifold. The existence of a line is a global condition on the manifold. In the case of the Ricci flow in dimension 3, under the assumption of nonnegative Ricci curvature, we can replace this global condition by a local curvature condition, namely the existence of a zero Ricci curvature. In higher dimensions, under the assumption of nonnegative curvature operator (recall Definition 3.10), although we may not have the splitting as a product, we obtain a certain rigidity which we describe below.

We motivate our discussion of the strong maximum principle by an example.

EXAMPLE 5.40 (Local nonuniqueness). *Consider* S^3 *with the following initial metric* g_0. *We take a unit cylinder* $[-1,1] \times S^2(1)$ *and cap the ends by unit hemispheres. We then smooth the metric near the two* S^2*'s where the metrics are glued to get a smooth metric* g_0 *on* S^3 *which on an open submanifold is isometric to a cylinder* $(-1/2, 1/2) \times S^2(1)$. *If we run the Ricci flow, then since* $\mathrm{Rm}(x_0, 0) > 0$ *for some* $x_0 \in S^3$, *we have* $\mathrm{Rm}(x,t) > 0$ *for all* $x \in S^3$ *and* $t > 0$. *Thus we have a local (incomplete) solution of the Ricci flow* $\left((-1/2, 1/2) \times S^2, g(t)\right)$, $t \in [0, \delta)$ *for some* $\delta > 0$, *where* $g(0)$ *is the standard unit cylinder and* $\mathrm{Rm}(g(t)) > 0$ *for* $t > 0$. *On the other hand, we also have the cylinder solution* $\left((-1/2, 1/2) \times S^2, \bar{g}(t)\right)$, *where* $\bar{g}(t)$ *is the product of* $(-1/2, 1/2)$ *with the 2-sphere of radius* $r(t) \doteqdot (1-2t)^{1/2}$ *(see (2.1)), which has the same initial metric.*

The above example shows that it is possible for the dimension of the image of the curvature operator to jump up. Similar examples can easily be constructed with initial metric $g(0)$ flat in an open set and $g(t)$ positively curved everywhere for $t > 0$.

Let $\operatorname{Im}(\mathrm{Rm}) \doteqdot \mathrm{Rm}\left(\wedge^2 T^* M^n\right)$ denote the **image** of the curvature operator. The **strong maximum principle for systems** as applied to the Riemann curvature operator under the Ricci flow says the following.

THEOREM 5.41 (Strong Maximum Principle for Rm). *Let* $(M^n, g(t))$, $t \in [0,T)$, *be a solution to the Ricci flow with nonnegative curvature operator:* $\mathrm{Rm}[g(t)] \geq 0$. *There exists* $\delta > 0$ *such that for each* $t \in (0,\delta)$, *the set* $\operatorname{Im}(\mathrm{Rm}[g(t)]) \subset \wedge^2 T^* M^n$ *is a smooth subbundle which is invariant under parallel translation and constant in time. Moreover,* $\operatorname{Im}(\mathrm{Rm}[g(x,t)])$ *is a Lie subalgebra of* $\wedge^2 T_x^* M^n \cong \mathfrak{so}(n)$ *for all* $x \in M$ *and* $t \in (0,\delta)$.

When M is a closed manifold, only at the initial time can the image of the curvature operator jump.

EXERCISE 5.42. *Show that if* M^n *is closed, then we may take* $\delta = T$.

HINT. See Theorem 5.54.

PROPOSITION 5.43. *In general, if we have a solution to the Ricci flow* $(M^n, g(t))$, $t \in [0,T]$, *(not necessarily complete) on a connected open manifold with* $\mathrm{Rm}[g(t)] \geq 0$, *then* $\operatorname{Im}(\mathrm{Rm}[g(t)]) \subset \wedge^2 T^* M^n$ *is a smooth subbundle which is invariant under parallel translation in space for all* $t \in (0,T]$. *There exist a finite sequence of times* $0 = t_0 < t_1 < t_2 < \cdots < t_k = T$ *such that* $\operatorname{Im}(\mathrm{Rm}[g(x,t)])$ *is a Lie subalgebra of* $\wedge^2 T_x^* M^n$ *independent of time for* $t \in (t_{i-1}, t_i]$, $i = 1, \ldots, k$. *Furthermore,* $\operatorname{Im}(\mathrm{Rm}[g(t_1)]) \subset \operatorname{Im}(\mathrm{Rm}[g(t_2)])$ *for* $t_1 < t_2$.

Suppose that $g(t)$, $t \in [0,T)$, is a solution of Ricci flow (not necessarily complete) with $\mathrm{Rm} \geq 0$. If $\mathrm{Rm}(x_1, 0) > 0$ for some $x_1 \in M$, then for all $x \in M$ and $t > 0$

$$\mathrm{Rm}(x,t) > 0.$$

If the solution is defined for $t > t_0$ where $t_0 < 0$, then $\mathrm{Rm}(x,t) > 0$ for all $x \in M$ and $t > t_0$.

If $\mathrm{Rm} = 0$ at some point (x_1, t_1), then $\mathrm{Rm} = 0$ for all $x \in M$ and $t \leq t_1$.

EXERCISE 5.44. *Construct a solution to Ricci flow* (M^n, g), $n \geq 3$, *with* $\operatorname{Im}(\mathrm{Rm}[g(x,t)]) \cong \mathfrak{so}(i+1)$ *for* $t \in (i-1, i]$, $i = 1, \ldots, n-1$.

Let $\left(M^2, g(t)\right)$, $t \in (\alpha, \omega)$, be a solution to the Ricci flow on a closed surface such that $R > 0$ at some point (x_0, t_0). The strong maximum principle implies $R(t) > 0$ for all $t \in (\alpha, \omega)$. The reason is if $R = 0$ at some (x_1, t_1), then $R(t) \equiv 0$ for $t < t_1$. By the uniqueness of solutions of the Ricci flow on closed manifolds, we would have $R(t) \equiv 0$ for all $t \in (\alpha, \omega)$, a contradiction.

If M^2 is noncompact, then there exists $\bar{t} \in [\alpha, \omega]$ such that $R \equiv 0$ for $t \leq \bar{t}$ and $R(x,t) > 0$ for $x \in M$ and $t > \bar{t}$.

Proof. Let

$$\mathcal{Z} \doteqdot \{t \in (\alpha, \omega) : R(g(t)) \equiv 0\}.$$

$\mathcal{Z}$ is closed so that $\bar{t} \doteqdot \sup\{t : t \in \mathcal{Z}\} \in \mathcal{Z}$. We have $\sup_M R(g(t)) > 0$ for all $t > \bar{t}$ and by the strong maximum principle $R \equiv 0$ for all $t \leq \bar{t}$. By the strong maximum principle it is also easy to see that $R(x,t) > 0$ for all $x \in M$ and $t > \bar{t}$.

5. 3-manifolds with nonnegative curvature

We now specialize to the case $n = 3$. Here things simplify considerably since the structure of the Lie algebra $\mathfrak{so}(3)$ is simple. The only Lie subalgebras are $\{0\}$, $\mathfrak{so}(3)$ and those isomorphic to $\mathfrak{so}(2)$ (i.e., the one dimensional subspaces of $\mathfrak{so}(3)$.) Suppose $t \in (0, \delta)$ and consider each of these cases for $\operatorname{Im}(\operatorname{Rm}[g(t)])$.

(1) (**Positive curvature**) If $\operatorname{Im}(\operatorname{Rm}[g(x,t)]) \cong \mathfrak{so}(3)$ for all x and $t > 0$, then the curvature operator is positive, that is, $(M^3, g(t))$ has positive sectional curvature for all $t > 0$. If M^3 is closed, then M^3 is diffeomorphic to a spherical space form by Hamilton's Theorem 3.6. If $(M^3, g(t))$ is complete and noncompact, then M^3 is diffeomorphic to euclidean space by Gromoll and Meyer [**240**].

(2) (**Flat**) If $\operatorname{Im}(\operatorname{Rm}[g(x,t)]) = \{0\}$, then $(M^3, g(t))$ is flat for all $t > 0$. If $(M^3, g(t))$ is complete, then such manifolds are classified by the **Bieberbach theorem** (see [**516**] or [**66**]).

(3) (**Splitting**) If $\operatorname{Im}(\operatorname{Rm}[g(x,t)]) \cong \mathfrak{so}(2)$, then by the theorem, there exists a parallel unit 2-form $\alpha \in \operatorname{Im}(\operatorname{Rm}[g(x,t)])$ which is independent of time. Taking first the Hodge star $*$ and then the metric dual $\natural$ of α, we obtain a unit vector field on M parallel with respect to each of the metrics $g(t)$

$$X \doteqdot (*\alpha)^{\natural}.$$

Now we can apply the **deRham local splitting theorem** to conclude that locally, $(M^3, g(t))$ is isometric to the metric product of a surface with positive curvature with the real line. If $(M^3, g(t))$ is complete, then the universal cover $\tilde{M}^3$, with the lifted family of metrics $\tilde{g}(t)$, is globally the product of the real line and a complete solution to the Ricci flow on a surface with positive curvature.

If M^3 is a closed manifold, we have the following.

Theorem 5.45 (Hamilton - closed 3-manifolds with $\operatorname{Rm} \geq 0$ or $\operatorname{Rc} \geq 0$). *If $(M^3, g(t))$, $t \in [0,T)$, is a solution to the Ricci flow on a closed 3-manifold with nonnegative sectional (Ricci) curvature, then for $t \in (0,T)$ the universal covering solution $\left(\tilde{M}^3, \tilde{g}(t)\right)$ is either:*

(1) $\mathbb{R}^3$ *with the standard flat metric,*

(2) *the product $(S^2, h(t)) \times \mathbb{R}$, where $h(t)$ is a solution to the Ricci flow with positive curvature, or*

(3) *has positive sectional (Ricci) curvature, and hence $\tilde{M}^3$ is diffeomorphic to S^3 (and M^3 is diffeomorphic to a spherical space form.)*

PROOF. **I.** If the sectional curvature of $g(t)$ is nonnegative, then since in dimension 3 this is equivalent to nonnegative curvature operator, we can apply Theorem 5.41 and the above classification of $\operatorname{Im}(\operatorname{Rm}[g(x,t)])$ to conclude the theorem.

II. Now we assume the Ricci curvature of $g(t)$ is nonnegative. The evolution of the Ricci tensor, in an evolving orthonormal frame, is given by adding $2\operatorname{Rc}^2$ to the RHS of (3.1):

$$\frac{\partial}{\partial t}R_{ab} = \Delta R_{ab} + Q_{ab}$$

where

$$Q_{ab} \doteqdot 3RR_{ab} - 4R_{ac}R_{cb} + \left(2\left|\operatorname{Rc}\right|^2 - R^2\right)\delta_{ab}.$$

Observe that if at a point R_{ab} is diagonal with eigenvalues A, B, C, then Q_{ab} is diagonal with eigenvalues

$$\begin{aligned}\alpha &= (B-C)^2 + A(B+C)\\ \beta &= (A-C)^2 + B(A+C)\\ \gamma &= (A-B)^2 + C(A+B).\end{aligned}$$

In particular, whenever $\operatorname{Rc} \geq 0$, we have $Q \geq 0$. Similar to Theorem 5.41, one can prove the following. If $\left(M^3, g(t)\right)$, $t \in [0,T)$, is a solution to the Ricci flow with nonnegative Ricci curvature, then there exists $\delta > 0$ such that for each $t \in (0,\delta)$, the set $\operatorname{Im}(\operatorname{Rc}[g(t)]) \subset TM^3$ is a smooth subbundle which is invariant under parallel translation and constant in time. If $\operatorname{Im}(\operatorname{Rc}[g(t)]) = TM^3$ for $t > 0$, then $g(t)$ has positive Ricci curvature and M^3 is diffeomorphic to a spherical space form. If $\operatorname{Im}(\operatorname{Rc}[g(t)]) \neq TM^3$, then locally there exists a parallel vector field, we we assume to be e_1 such that $A = \operatorname{Rc}(e_1,e_1) = 0$. This implies $Q(e_1,e_1) = \alpha = 0$ so that $B = C$. If $B = C = 0$, then the solution is flat locally and hence globally. On the other hand if $B = C > 0$, then the solution has nonnegative sectional curvature and we can apply the previous arguments to see that the universal cover splits as the product of a surface with a line. □

In the complete noncompact case W.-X. Shi [**461**] has obtained the same classification. This relies on the same arguments as in the compact case plus the additional results (due to W.-X. Shi [**462**], [**463**]) that given a complete 3-manifold with bounded nonnegative Ricci curvature, there exists a solution to the Ricci flow with bounded nonnegative Ricci curvature. We remind the reader that strong maximum principle arguments are local.

6. Manifolds with nonnegative curvature operator

Recall in dimension 3, Hamilton's 1982 theorem states that the normalized Ricci flow evolves a metric with positive Ricci curvature on a closed 3-manifold to a constant positive sectional curvature metric. In [**257**] Hamilton established the long time existence and convergence (exponentially fast in every C^k norm) of the Ricci flow on closed 4-manifolds with positive curvature operator to a metric with constant positive sectional curvature(see Theorem 3.16 above). In particular, such manifolds are diffeomorphic to either S^4 or $\mathbb{R}P^4$. Hamilton's proof uses the maximum principle and a convex analysis of the system of ODE (3.21) corresponding to the evolution equation for Rm to prove pinching estimates for the curvature operator. One can extend this classification to 4-manifolds with nonnegative curvature [**257**].

THEOREM 5.46 (4–manifolds with $\mathrm{Rm} \geq 0$). *If $\left(M^4, g(t)\right)$, $t \in [0,T)$, is a solution to the Ricci flow on a closed 4-manifold with nonnegative curvature operator, then for $t \in (0,T)$ the universal covering solution $\left(\tilde{M}^4, \tilde{g}(t)\right)$ is either:*

1. *$\mathbb{R}^4$ with the standard flat metric,*
2. *the product $\left(S^3, h(t)\right) \times \mathbb{R}$, where $h(t)$ is a solution to the Ricci flow with positive sectional curvature,*
3. *the product $\left(S^2, h_1(t)\right) \times \left(S^2, h_2(t)\right)$, where both $h_1(t)$ and $h_2(t)$ are solutions to the Ricci flow with positive curvature,*
4. *the product $\left(S^2, h_1(t)\right) \times \left(\mathbb{R}^2, h_2\right)$, where $h_1(t)$ is a solution to the Ricci flow with positive curvature and h_2 is the standard flat metric,*
5. *$\mathbb{C}P^2$ with a solution to the Ricci flow whose curvature operators are positive on subspace of real $(1,1)$-forms in the space of 2-forms, or*
6. *has positive curvature operator, and hence $\tilde{M}^4$ is diffeomorphic to S^4 (M^4 is diffeomorphic to either S^4 or $\mathbb{R}P^4$.)*

PROOF. In each case the Lie subalgebra $\mathrm{Im}(\mathrm{Rm})$ of $\wedge^2 \cong \mathfrak{so}(4)$ is given as follows. 1. $\{0\}$, 2. $\mathfrak{so}(3)$, 3. $\mathfrak{so}(2) \times \mathfrak{so}(2)$, 4. $\mathfrak{so}(2)$, 5. $\mathfrak{u}(2) = \mathfrak{so}(3) \times \mathfrak{so}(2)$, 6. $\mathfrak{so}(4)$. □

In all dimensions at least 4, Huisken [**287**], Margerin [**364**], [**365**], and Nishikawa [**407**], [**408**] proved that if the initial metric has sufficiently pointwise pinched positive sectional curvatures, then the solution to the normalized flow converges to a constant positive sectional curvature metric. The main curvature estimates in these papers are based on considering scalar quantities measuring pointwise pinching of the curvatures (difference from constant positive sectional curvature).

CONJECTURE 5.47 (Hamilton - Closed manifolds with positive curvature operator). *Given a closed Riemannian manifold (M^n, g_0), $n \geq 5$, with positive curvature operator, the solution to the normalized Ricci flow exists for all time and converges to a metric of constant positive sectional curvature.*

One difficulty in studying this conjecture is to obtain some sort of 'curvature pinching is preserved' type estimate. By the following result [**486**], to prove convergence to a constant positive sectional curvature metric, it is sufficient to prove convergence to an Einstein metric with positive curvature operator. The presentation below follows section 7.5 of [**423**].

THEOREM 5.48 (Tachibana 1974 - Einstein manifolds with $\mathrm{Rm} \geq 0$).

(1) *If* (M^n, g) *is an Einstein manifold with positive curvature operator, then* g *has constant positive sectional curvature.*
(2) *If* (M^n, g) *is an Einstein manifold with nonnegative curvature operator, then* g *is locally symmetric.*

In other words, the fixed points of the normalized Ricci flow with positive curvature operator have constant positive sectional curvature.

PROBLEM 5.49. *Is there an analogue of Tachibana's theorem for shrinking gradient Ricci solitons with positive curvature operator on closed manifolds?*

By Hamilton's 1986 theorem, this is true when $n = 4$. If $n = 3$, then Hamilton's 1982 theorem implies the stronger result that there are no shrinking gradient Ricci solitons with positive Ricci curvature on closed 3-manifolds. When $n = 2$, the Kazdan-Warner identity (see Theorem 6.2) yields the desired result. In the Kähler analogue of the above problem one assumes the bisectional curvature is positive. Here, by the convergence result of Chen and Tian [**121**], [**122**], a Kähler-Ricci soliton is Kähler-Einstein and hence the Fubini-Study metric.

REMARK 5.50. *Mok and Zhong* [**383**] *proved that a simply connected Kähler-Einstein manifold with nonnegative bisectional curvature is symmetric. Earlier, Matsushima* [**366**] *proved that a simply connected Kähler-Einstein manifold with nonnegative curvature operator (when* $\dim_{\mathbb{C}} M = 2$, *he only required nonnegative bisectional curvature) is symmetric.*

We begin with a nice identity due to Lichnerowicz [**347**].

LEMMA 5.51 (Lichnerowicz 1958). *On any closed Riemannian manifold, the* L^2*-norms of the divergence and covariant derivatives of* Rm *are related to their commutator* $[\operatorname{div}, \nabla]$ *by:*

$$\int_{M^n} |\operatorname{div}(\operatorname{Rm})|^2 \, d\mu - \int_{M^n} \left\langle \operatorname{Rm}^\flat, \operatorname{div}(\nabla \operatorname{Rm}) - \nabla \operatorname{div}(\operatorname{Rm}) \right\rangle d\mu = \frac{1}{2} \int_{M^n} |\nabla \operatorname{Rm}|^2 \, d\mu, \tag{5.20}$$

where $\left(\operatorname{Rm}^\flat\right)_{ijk\ell} = R_{\ell ijk}$, $\operatorname{div}(\operatorname{Rm})_{ijk} \doteqdot \nabla_\ell R_{ijk\ell}$ *and* $\operatorname{div}(\nabla \operatorname{Rm})_{ijk\ell} \doteqdot \nabla_m \nabla_i R_{jk\ell m}$.

PROOF. Note that $\operatorname{div}(\nabla \operatorname{Rm}) = \Delta \operatorname{Rm}$. From the second Bianchi identity

$$\begin{aligned} R_{ijk\ell} \nabla_m \nabla_m R_{ijk\ell} &= -R_{ijk\ell} \nabla_m \nabla_i R_{jmk\ell} - R_{ijk\ell} \nabla_m \nabla_j R_{mik\ell} \\ &= 2 R_{ijk\ell} \nabla_m \nabla_j R_{imk\ell}, \end{aligned} \tag{5.21}$$

using the fact that $R_{ijk\ell}$ is anti-symmetric in i and j. Integrating the identity (5.21) by parts, we have

$$\begin{aligned}
0 &= \int_{M^n} \left(|\nabla \operatorname{Rm}|^2 + 2R_{ijk\ell}\nabla_m \nabla_j R_{imk\ell} \right) d\mu \\
&= \int_{M^n} \left(|\nabla \operatorname{Rm}|^2 + 2R_{ijk\ell}\left(\nabla_m \nabla_j R_{imk\ell} - \nabla_j \nabla_m R_{imk\ell}\right) - 2\left|\operatorname{div}(\operatorname{Rm})\right|^2 \right) d\mu \\
&= \int_{M^n} \left(|\nabla \operatorname{Rm}|^2 + 2R_{ijk\ell}\left(\operatorname{div}(\nabla \operatorname{Rm})_{jk\ell i} - \nabla_j \operatorname{div}(\operatorname{Rm})_{k\ell i}\right) - 2\left|\operatorname{div}(\operatorname{Rm})\right|^2 \right) d\mu.
\end{aligned}$$

□

The following result gives one hope that the middle term in (5.20) has a sign.

LEMMA 5.52 (Berger). *If (M^n, g) is a Riemannian manifold with nonnegative sectional curvature and if α is a symmetric 2-tensor, then*

$$\langle \alpha, \operatorname{div}(\nabla \alpha) - \nabla \operatorname{div}(\alpha) \rangle \geq 0.$$

Since

$$(\operatorname{div}(\nabla\alpha) - \nabla \operatorname{div}(\alpha))_{ij} = \nabla_k \nabla_i \alpha_{jk} - \nabla_i \nabla_k \alpha_{jk} = -R_{kij\ell}\alpha_{\ell k} + R_{i\ell}\alpha_{j\ell},$$

we have

$$\langle \alpha, \operatorname{div}(\nabla \alpha) - \nabla \operatorname{div}(\alpha) \rangle = -R_{kij\ell}\alpha_{\ell k}\alpha_{ij} + R_{i\ell}\alpha_{j\ell}\alpha_{ij}. \tag{5.22}$$

Choose an orthonormal frame $\{e_a\}_{a=1}^n$ such that $\alpha = \sum_{a=1}^n \lambda_a e_a \otimes e_a$, $\lambda_a \in \mathbb{R}$. We may rewrite (5.22) as

$$\begin{aligned}
\langle \alpha, \operatorname{div}(\nabla \alpha) - \nabla \operatorname{div}(\alpha) \rangle &= -\sum_{a,b=1}^{n} \left(\langle R(e_a, e_b) e_b, e_a \rangle \lambda_a \lambda_b + \langle R(e_a, e_b) e_b, e_a \rangle \lambda_a^2 \right) \\
&= \sum_{a<b} \langle R(e_a, e_b) e_b, e_a \rangle (\lambda_a - \lambda_b)^2 \geq 0.
\end{aligned}$$

Indeed our hope is justified and we have the following.

LEMMA 5.53 (Tachibana 1974). *If (M^n, g) has nonnegative curvature operator and $T : \wedge^2 M^n \to \wedge^2 M^n$ is self-adjoint, then*

$$K \doteqdot \left\langle T^\flat, \operatorname{div}(\nabla T) - \nabla \operatorname{div}(T) \right\rangle \geq 0$$

where $\left(T^\flat\right)_{ijk\ell} \doteqdot T_{\ell ijk}$.

PROOF. We have

$$K = T_{\ell ijk}\left(\nabla_m \nabla_i T_{\ell mjk} - \nabla_i \nabla_m T_{\ell mjk}\right)$$

and

$$\nabla_m \nabla_i T_{\ell mjk} - \nabla_i \nabla_m T_{\ell mjk} = R_{im\ell p}T_{pmjk} + R_{immp}T_{\ell pjk} + R_{imjp}T_{\ell mpk} + R_{imkp}T_{\ell mjp}$$

so that
(5.23)

$$K = \underbrace{R_{immp}T_{\ell pjk}T_{\ell ijk}}_{(1)} + \underbrace{R_{imkp}T_{\ell mjp}T_{\ell ijk}}_{(2)} + \underbrace{R_{imjp}T_{\ell mpk}T_{\ell ijk}}_{(3)} + \underbrace{R_{im\ell p}T_{pmjk}T_{\ell ijk}}_{(4)}.$$

We may also consider the $(4,0)$-tensor T as a $(3,1)$-tensor so that given vectors X, Y, Z, $T(X,Y,Z)$ is also a vector. Note that $e_a \wedge T(e_b, e_c, e_d) = T_{bcdp} e_a \wedge e_p$. Given an orthonormal frame $\{e_a\}_{a=1}^n$, define the $(4,2)$-tensor θ by:

$$\begin{aligned}\theta_{abcd} &\doteqdot \underbrace{e_a \wedge T(e_b, e_c, e_d)}_{A} + \underbrace{e_b \wedge T(e_a, e_d, e_c)}_{B} \\ &+ \underbrace{e_c \wedge T(e_d, e_a, e_b)}_{C} + \underbrace{e_d \wedge T(e_c, e_b, e_a)}_{D} \\ &\in \wedge^2 M^n.\end{aligned}$$

We claim

$$4K = \langle \mathrm{Rm}(\theta_{abcd}), \theta_{abcd} \rangle, \tag{5.24}$$

which is nonnegative since $\mathrm{Rm} \geq 0$. First

$$\begin{aligned}\langle \mathrm{Rm}(A), A \rangle &= \langle \mathrm{Rm}(e_a \wedge T(e_b, e_c, e_d)), e_a \wedge T(e_b, e_c, e_d) \rangle \\ &= R_{apqa} T_{bcdp} T_{bcdq} = R_{immp} T_{\ell pjk} T_{\ell ijk} = (1),\end{aligned}$$

where we used $\langle \mathrm{Rm}(e_a \wedge e_p), e_a \wedge e_q \rangle = R_{apqa}$ and then relabelled indices to get the last equality. Second

$$\begin{aligned}\langle \mathrm{Rm}(A), B \rangle &= \langle \mathrm{Rm}(e_a \wedge T(e_b, e_c, e_d)), e_b \wedge T(e_a, e_d, e_c) \rangle \\ &= R_{apqb} T_{bcdp} T_{adcq} = R_{imkp} T_{\ell mjp} T_{\ell ijk} = (2),\end{aligned}$$

using the symmetries of Rm and T. Third

$$\begin{aligned}\langle \mathrm{Rm}(A), C \rangle &= \langle \mathrm{Rm}(e_a \wedge T(e_b, e_c, e_d)), e_c \wedge T(e_d, e_a, e_b) \rangle \\ &= R_{apqc} T_{bcdp} T_{dabq} = R_{imjp} T_{\ell mpk} T_{\ell ijk} = (3).\end{aligned}$$

Fourth, and finally

$$\begin{aligned}\langle \mathrm{Rm}(A), D \rangle &= \langle \mathrm{Rm}(e_a \wedge T(e_b, e_c, e_d)), e_d \wedge T(e_c, e_b, e_a) \rangle \\ &= R_{apqd} T_{bcdp} T_{cbaq} = R_{im\ell p} T_{pmjk} T_{\ell ijk} = (4).\end{aligned}$$

From the above four formulas and (5.23), we can easily verify (5.24) since using the symmetries of the tensors in consideration, we have

$$\begin{aligned}\frac{1}{4} \langle \mathrm{Rm}(\theta_{abcd}), \theta_{abcd} \rangle &= \langle \mathrm{Rm}(A), A \rangle + \langle \mathrm{Rm}(A), B \rangle \\ &+ \langle \mathrm{Rm}(A), C \rangle + \langle \mathrm{Rm}(A), D \rangle \\ &= (1) + (2) + (3) + (4) = K.\end{aligned}$$

□

Putting the above together, we get

PROOF OF THEOREM 5.48. First we assume $\mathrm{Rm} \geq 0$. Since $\operatorname{div}(\mathrm{Rm})_{ijk} = \nabla_i R_{jk} - \nabla_j R_{ik}$, if a metric is Einstein, then $\operatorname{div}(\mathrm{Rm}) = 0$. Hence (5.20) and Lemma 5.53 (in particular, (5.24)) tell us

$$\begin{aligned} 0 &= \frac{1}{2}\int_{M^n} |\nabla \mathrm{Rm}|^2 \, d\mu + \int_{M^n} \left\langle \mathrm{Rm}^\flat, \operatorname{div}(\nabla \mathrm{Rm}) - \nabla \operatorname{div}(\mathrm{Rm}) \right\rangle d\mu \\ &= \frac{1}{2}\int_{M^n} |\nabla \mathrm{Rm}|^2 \, d\mu + \frac{1}{4}\int_{M^n} \left\langle \mathrm{Rm}\left(\tilde{\theta}_{abcd}\right), \tilde{\theta}_{abcd} \right\rangle d\mu, \end{aligned} \tag{5.25}$$

where

$$\begin{aligned} \tilde{\theta}_{abcd} \doteqdot {} & e_a \wedge \mathrm{Rm}(e_b, e_c) e_d + e_b \wedge \mathrm{Rm}(e_a, e_d) e_c \\ & + e_c \wedge \mathrm{Rm}(e_d, e_a) e_b + e_d \wedge \mathrm{Rm}(e_c, e_b) e_a. \end{aligned} \tag{5.26}$$

Since both terms on the RHS of (5.25) are nonnegative, they are zero. This proves $\nabla \mathrm{Rm} \equiv 0$ and hence part 2. Now we assume $\mathrm{Rm} > 0$, so that we have $\tilde{\theta}_{abcd} = 0$ for all a, b, c, d. Part 1 of the theorem follows from the fact that $\tilde{\theta} = 0$ implies (M^n, g) has constant sectional curvature.

To see this we multiply (5.26) by V^a, W^b, X^c, Y^d and sum to get:

$$V \wedge \mathrm{Rm}(W, X) Y + W \wedge \mathrm{Rm}(V, Y) X + X \wedge \mathrm{Rm}(Y, V) W + Y \wedge \mathrm{Rm}(X, W) V = 0 \tag{5.27}$$

for all vectors V, W, X, Y ($V = V^a e_a$, etc.). Suppose V, W and $X = Y$ are orthonormal. By considering the $V \wedge W$ component of the above expression, we see that $K(X \wedge V) = K(X \wedge W)$. This implies that if X is unit, then $K(X \wedge W) = \frac{1}{n-1}\mathrm{Rc}(X, X) = \frac{R}{n(n-1)}$ is constant on M^n. □

By applying Hamilton's strong maximum principle for systems, we can classify those Riemannian manifolds with nonnegative curvature operator which do not have positive curvature operator. (See Gallot-Meyer [**219**], Cao-Chow [**77**], and Chow-Yang [**161**].)

THEOREM 5.54 (Closed manifolds with nonnegative curvature operator). *If (M^n, g) is a closed Riemannian manifold with nonnegative curvature operator, then its universal covering manifold $\left(\tilde{M}^n, \tilde{g}\right)$ is isometric to the product of the following:*

(1) *euclidean space,*
(2) *closed symmetric space,*
(3) *closed Riemannian manifold with positive curvature operator,*
(4) *closed Kähler manifold with positive curvature operator on real $(1,1)$-forms.*

The two main ingredients in the proof of the above theorem is Hamilton's strong maximum principle for tensors and **Berger's holonomy classification theorem** [**44**].

THEOREM 5.55 (Berger). *If (M^n, g) is a simply connected, irreducible Riemannian manifold, then either (M^n, g) is a symmetric space of rank ≥ 2, or the holonomy group is one of the following:*

(1) SO(n)
(2) U$(n/2)$
(3) SU$(n/2)$
(4) Sp$(n/4)$ Sp(1)
(5) Sp$(n/4)$
(6) G_2 *($n = 7$)*
(7) Spin(7) *($n = 8$)*
(8) Spin(9) *($n = 16$)*.

Proof of Theorem 5.54. By the short time existence Theorem 2.48 and since nonnegative curvature operator is preserved (Corollary 3.18), there exists a solution $g(t)$ of the Ricci flow with $g(0) = g$ and nonnegative curvature operator on some time interval $t \in [0, \varepsilon)$. By Hamilton's strong maximum principle for systems (Theorem 5.41), for $t > 0$, the image of the curvature operator Im (Rm) is invariant under parallel translation and constant in time, and is a Lie subalgebra of $\wedge^2 \cong \mathfrak{so}(n)$ isomorphic to the holonomy Lie algebra $\mathfrak{h}$. By Berger's theorem, if $\left(\tilde{M}^n, \tilde{g}(t)\right)$ is not a symmetric space of rank ≥ 2, then the holonomy Lie algebra $\mathfrak{h}$ is one of the 8 listed above. If the holonomy group $\mathfrak{H}$ is either SU$(n/2)$, Sp$(n/4)$, Spin(7) or G_2, we obtain a contradiction since the metrics in these cases are Ricci flat, and hence flat (since they have nonnegative curvature operator). If $\mathfrak{H} \cong$ Spin(9), then Alekseevskii proved that $\left(\tilde{M}^n, \tilde{g}(t)\right)$ is symmetric and either the Cayley (octonian) projective plane or the dual Cayley disk [**4**]. Hence $\mathfrak{H}$ is isomorphic to either SO(n), U$(n/2)$ or Sp$(n/4)$ Sp(1) (this was observed by Alekseevskii). In the first case, $\mathfrak{h} \cong \mathfrak{so}(n)$ and hence Im (Rm) $= \wedge^2$, which implies the curvature operator is positive: case 3. In the second case, $\left(\tilde{M}^n, \tilde{g}(t)\right)$ is a closed Kähler manifold with Im (Rm) $\cong \mathfrak{h} \cong \mathfrak{u}(n/2)$. This implies the curvature operator is positive on real $(1,1)$-forms: case 4. In the last case, where $\mathfrak{H} \cong$ Sp$(n/4)$ Sp(1), we have $\left(\tilde{M}^n, \tilde{g}(t)\right)$ is Einstein. By Tachibana's Theorem 5.48, we conclude $\left(\tilde{M}^n, \tilde{g}(t)\right)$ is locally symmetric and hence symmetric since it is simply connected. □

In case 3, when the dimension is at least 4, by the result of Micallef-Moore [**375**] (which assumes a curvature condition weaker than either positive curvature operator or 1/4-pinching), the factor is homeomorphic to a sphere. In dimension 3 or 4, by Hamilton's results, the factor is diffeomorphic to a sphere.

In case 4, the factor has positive bisectional curvature (which is a weaker condition than positive curvature operator on real $(1,1)$-forms). Andreotti-Frankel [**212**] ($n = 2$), Kobayashi-Ochiai [**328**] and Mabuchi [**362**] ($n = 3$), and Mori [**388**] and Siu-Yau [**476**] (all dimensions) showed that closed Kähler manifolds with positive bisectional curvature are biholomorphic to $\mathbb{C}P^n$ (Mabuchi and Mori's results are algebraic and more general). See

Bando [**37**] and Mok [**382**] for the classification of closed Kähler manifolds with nonnegative bisectional curvature.

7. Notes and commentary

§1. If initial metric has bounds on some derivatives of curvature, we can improve the local estimates for the higher derivatives as follows.

Theorem 5.56 (Higher derivative estimates assuming bounds on some derivatives). *For any $\alpha, K, K_\ell, r, \ell \geq 0$, n and $m \in \mathbb{N}$, there exists a constant $C < \infty$ depending only on $\alpha, K, K_\ell, r, \ell, n$ and m such that if M^n is a manifold, $p \in M^n$, and $g(t)$, $t \in [0, \tau]$, where $0 < \tau \leq \alpha/K$, is a solution to the Ricci flow on an open neighborhood U of p containing $\bar{B}_{g(0)}(p, r)$ as a compact subset, and if*

$$|\operatorname{Rm}(x,t)| \leq K \text{ for all } x \in U \text{ and } t \in [0,\tau],$$
$$\left|\nabla^k \operatorname{Rm}(x,0)\right| \leq K_\ell \text{ for all } x \in U \text{ and } k \leq \ell$$

then

$$|\nabla^m \operatorname{Rm}(y,t)| \leq \frac{C}{t^{\max\{m-\ell,0\}/2}}$$

for all $y \in B_{g(0)}(p, r/2)$ and $t \in (0, \tau]$.

§6. (Problem 5.49.) It is not clear in the case of shrinking gradient solitons with positive curvature operator whether an argument along the lines of Tachibana's will work to show that such metrics have constant sectional curvature. For example, if $R_{ij} + \nabla_i \nabla_j f + \varepsilon g_{ij} = 0$, then

$$\nabla_i R_{ijk\ell} = -\nabla_k R_{j\ell} + \nabla_\ell R_{jk} = \nabla_k \nabla_\ell \nabla_j f - \nabla_\ell \nabla_k \nabla_j f = -R_{k\ell ji} \nabla_i f = R_{ijk\ell} \nabla_i f,$$

that is

$$\nabla_i \left(e^{-f} R_{ijk\ell} \right) = 0.$$

Tracing this we also see that $\nabla_i \left(e^{-f} R_{i\ell} \right) = 0$. By the second Bianchi identity we have

$$\begin{aligned} &\nabla_i \left(e^{-f} R_{jk\ell m} \right) + \nabla_j \left(e^{-f} R_{ki\ell m} \right) + \nabla_k \left(e^{-f} R_{ij\ell m} \right) \\ &= -\left(e^{-f} R_{jk\ell m} \nabla_i f + e^{-f} R_{ki\ell m} \nabla_j f + e^{-f} R_{ij\ell m} \nabla_k f \right). \end{aligned}$$

This implies

$$\begin{aligned}\Delta\left(e^{-f}R_{jk\ell m}\right) &= -\nabla_i\nabla_j\left(e^{-f}R_{ki\ell m}\right)-\nabla_i\nabla_k\left(e^{-f}R_{ij\ell m}\right)\\ &\quad-\nabla_i\left(e^{-f}R_{jk\ell m}\nabla_i f+e^{-f}R_{ki\ell m}\nabla_j f+e^{-f}R_{ij\ell m}\nabla_k f\right)\\ &=[\nabla_j,\nabla_i]\left(e^{-f}R_{ki\ell m}\right)+[\nabla_k,\nabla_i]\left(e^{-f}R_{ij\ell m}\right)\\ &\quad-\left(e^{-f}R_{jk\ell m}\Delta f+e^{-f}R_{ki\ell m}\nabla_i\nabla_j f+e^{-f}R_{ij\ell m}\nabla_i\nabla_k f\right)\\ &\quad-\nabla_i f\nabla_i\left(e^{-f}R_{jk\ell m}\right).\end{aligned}$$

Hence

$$\begin{aligned}&\Delta\left(e^{-f}R_{jk\ell m}\right)+\nabla_i f\nabla_i\left(e^{-f}R_{jk\ell m}\right)\\ &=[\nabla_j,\nabla_i]\left(e^{-f}R_{ki\ell m}\right)+[\nabla_k,\nabla_i]\left(e^{-f}R_{ij\ell m}\right)\\ &+e^{-f}R_{jk\ell m}\left(R+n\varepsilon\right)+e^{-f}R_{ki\ell m}\left(R_{ij}+\varepsilon g_{ij}\right)+e^{-f}R_{ij\ell m}\left(R_{ik}+\varepsilon g_{ik}\right).\end{aligned}$$

That is

$$\begin{aligned}&\operatorname{div}\left(e^{f}\nabla\left(e^{-f}R_{jk\ell m}\right)\right)\\ &=R_{ijkp}R_{pi\ell m}+R_{ij\ell p}R_{kipm}+R_{ijmp}R_{ki\ell p}\\ &+R_{ikjp}R_{ip\ell m}+R_{ik\ell p}R_{ijpm}+R_{ikmp}R_{ij\ell p}\\ &+R_{jk\ell m}R+(n-2)\,\varepsilon R_{jk\ell m}.\end{aligned}$$

Integrating this against $e^{-f}R_{jk\ell m}$ yields

$$-\int_{M^n}\left|\nabla\left(e^{-f}R_{jk\ell m}\right)\right|^2 e^f d\mu=\int_{M^n}Je^{-f}d\mu \tag{5.28}$$

where

$$\begin{aligned}J\doteqdot{}&R_{jk\ell m}\left(R_{ijkp}R_{pi\ell m}+R_{ij\ell p}R_{kipm}+R_{ijmp}R_{ki\ell p}\right)\\ &+R_{jk\ell m}\left(R_{ikjp}R_{ip\ell m}+R_{ik\ell p}R_{ijpm}+R_{ikmp}R_{ij\ell p}\right)\\ &+\left(R+(n-2)\,\varepsilon\right)\left|R_{jk\ell m}\right|^2.\end{aligned}$$

However, when f is nonzero, it is not clear that the RHS of (5.28) has a sign.

CHAPTER 6

Some miscellaneous techniques for the Ricci, Yamabe and cross curvature flows

A number of the ideas and techniques on the Riemannian geometry of surfaces (may) generalize to higher dimensions. In this chapter we take a moment to consider some of these ideas which will not be discussed later in the book. We begin by recalling a formula of Bourguignon-Ezin for conformal vector fields which implies an expanding or steady Yamabe soliton has constant scalar curvature. Then we consider Ben Andrews' n-dimensional Poincaré type inequality which in dimension 2 is related to Hamilton's entropy monotonicity. It would be interesting to find a higher dimensional application of Andrews' inequality. In §3 we observe the gradient of Hamilton's entropy is the matrix Harnack quantity. This relation between entropy and Harnack has an analogue for Perelman's entropy functional as we shall see in Volume 2. In §4 we consider the Aleksandrov reflection technique applied to the Yamabe flow. It would interesting if this technique could be applied to ancient or immortal solutions. Another interesting technique for the Ricci flow on surfaces is Hamilton's isoperimetric constant monotonicity formula we consider in §5. Finally, in the last section we summarize some results about the cross curvature flow of three-manifolds with sectional curvature having a sign. This nonlinear curvature flow of metrics is in a sense dual to the Ricci flow.

1. Kazdan-Warner type identities and Yamabe and Ricci solitons

Recall the following.

DEFINITION 6.1. *A vector field X is a **conformal Killing vector field** if*

$$\nabla_i X_j + \nabla_j X_i = \frac{2}{n} \operatorname{div}(X)\, g_{ij}. \tag{6.1}$$

On the 2-sphere we have the following identity.

THEOREM 6.2 (Kazdan-Warner, [**313**]). *If X is a conformal Killing vector field, then*

$$\int_{S^2} \langle \nabla R, X \rangle \, d\mu = \int_{S^2} R \operatorname{div} X \, d\mu = 0.$$

EXERCISE 6.3. *Show that Theorem 6.2 implies any Ricci soliton on S^2 has constant curvature.*

The same is true when $\chi\left(M^2\right) < 0$. In Volume 2 we shall see this following from an estimate for solutions of the Ricci flow on surfaces with $\chi < 0$ (see also [**258**] or §5.5 of [**153**].)

In higher dimensions we have the following result.

PROPOSITION 6.4 (Rc < 0 implies no conformal Killing vector fields.). *If (M^n, g) is a closed Riemannian manifold with* $\mathrm{Rc} < 0$, *then there are no nonzero conformal Killing vector fields.*

PROOF. Taking divergence of the equation (6.1) we have

$$\Delta X_j + R_{jk} X_k + \left(1 - \frac{2}{n}\right) \nabla_j \operatorname{div}(X) = 0.$$

Hence

$$\begin{aligned}\frac{1}{2}\Delta |X|^2 &= |\nabla X|^2 + \langle \Delta X, X\rangle \\ &= |\nabla X|^2 - \mathrm{Rc}(X,X) - \left(1-\frac{2}{n}\right)\langle \nabla \operatorname{div}(X), X\rangle .\end{aligned}$$

Integrating this by parts, we obtain

$$0 = \int_{M^n} \left(|\nabla X|^2 - \mathrm{Rc}(X,X) + \left(1 - \frac{2}{n}\right) |\operatorname{div}(X)|^2 \right) d\mu.$$

Since each of the three terms on the RHS are nonnegative and $\mathrm{Rc} < 0$, we conclude $X = 0$. □

On a surface, in each conformal class, there exists a constant curvature metric. Since the property of being a conformal Killing vector field is a conformal invariant, we have another proof of the following.

COROLLARY 6.5 ($\chi < 0$ surface has no conformal Killing vector fields.). *If $\left(M^2, g\right)$ is a closed surface with $r < 0$, then there are no nonzero conformal Killing vector fields. In particular, all Ricci solitons have constant curvature.*

Note that in all dimensions, we have the following result, which extends the Kazdan-Warner identity.

PROPOSITION 6.6 (Bourguignon-Ezin 1987, [**54**]). *If (M^n, g) is a closed Riemannian manifold with $n \geq 3$ and if X is a conformal Killing vector field, then*

$$\int_{M^n} \langle \nabla R, X\rangle \, d\mu = \int_{M^n} R \operatorname{div} X \, d\mu = 0.$$

PROOF. (1) Taking the inner product of equation (6.1) with R_{ij}, integrating by parts, and applying the contracted second Bianchi identity, we have

$$\begin{aligned}\frac{1}{n}\int_{M^n} R \operatorname{div}(X)\, d\mu &= \int_{M^n} R_{ij} \nabla_i X_j d\mu = -\int_{M^n} \langle \operatorname{div}(\mathrm{Rc}), X\rangle \, d\mu \\ &= -\frac{1}{2}\int_{M^n} \langle \nabla R, X\rangle \, d\mu = \frac{1}{2}\int_{M^n} R \operatorname{div}(X)\, d\mu.\end{aligned}$$

Hence if $n \neq 2$, we obtain $\int_{M^n} R \operatorname{div}(X) \, d\mu = 0$.

(2) A slightly different proof is as follows. Note that for any 2-tensor a_{ij} (not necessarily symmetric)

$$\nabla_i \nabla_j a_{ij} - \nabla_j \nabla_i a_{ij} = R_{jk} a_{kj} - R_{ik} a_{ik} = 0.$$

Hence $\nabla_i \nabla_j \nabla_i X_j = \nabla_i \nabla_j \nabla_j X_i$, so that by taking the double divergence of (6.1) and using the contracted second Bianchi identity, we have

$$\begin{aligned} \frac{1}{n} \Delta \operatorname{div}(X) = \nabla_i \nabla_j \nabla_i X_j &= \Delta \operatorname{div}(X) + \nabla_i (R_{ik} X_k) \\ &= \Delta \operatorname{div}(X) + \frac{1}{2} \langle \nabla R, X \rangle + R_{ik} \nabla_i X_k. \end{aligned}$$

Substituting (6.1) into this yields (see also [**347**], p. 134)

$$\Delta (\operatorname{div} X) + \frac{1}{n-1} R \operatorname{div} X + \frac{n}{2(n-1)} \langle \nabla R, X \rangle = 0$$

Integrating this implies the proposition. □

As a consequence, we have the following.

Proposition 6.7 (Expanding or steady Yamabe solitons have R constant). *An expanding or steady* ***Yamabe soliton,*** *that is, a metric g with*

$$(r - R) g_{ij} = \nabla_i X_j + \nabla_j X_i,$$

for some vector field X and where $r \leq 0$, on a closed n-dimensional manifold has constant scalar curvature.

Exercise 6.8. *Prove the above proposition.*

Solution. We compute

$$\begin{aligned} \int_{M^n} (r - R)^2 \, d\mu &= -\int_{M^n} (r - R) g_{ij} \cdot R_{ij} d\mu \\ &= -2 \int_{M^n} \nabla_i X_j R_{ij} d\mu \\ &= \int_{M^n} X_j \nabla_j R d\mu = 0. \end{aligned}$$

For the Ricci flow we have:

Proposition 6.9 (Expanding or steady Ricci solitons are Einstein). *Any expanding or steady Ricci soliton on a closed n-dimensional manifold is Einstein.*

Proof. Recall that if g is an expanding or steady Ricci soliton on a closed manifold, then there exists a vector field X and $\lambda \geq 0$ such that

$$R_{ij} + \frac{1}{2} (\nabla_i X_j + \nabla_j X_i) - \frac{r}{n} g_{ij} = 0.$$

Similar to (8.17), since $\frac{\partial}{\partial t}R = \mathcal{L}_X R + \frac{2}{n}rR$, we have

$$0 = \Delta(R-r) - \langle X, \nabla(R-r)\rangle + 2\left|\operatorname{Rc} - \frac{r}{n}g\right|^2 + \frac{2}{n}r(R-r), \tag{6.2}$$

where $r \leq 0$. At any point $x_0 \in M^n$ such that $R(x_0) = R_{\min}$, we have

$$2\left|\operatorname{Rc} - \frac{r}{n}g\right|^2 + \frac{2}{n}r(R-r) \leq 0.$$

Since both r and $R(x_0) - r$ are non-positive, this implies $\left|\operatorname{Rc} - \frac{r}{n}g\right|^2(x_0) = 0$. Tracing this implies $R_{\min} = R(x_0) = r$, and hence $R \equiv r$. Substituting this back into (6.2), we conclude that $\operatorname{Rc} \equiv \frac{r}{n}g$. □

2. Andrews' Poincaré type inequality

Recall **Hamilton's surface entropy** (see [**258**]). N is defined for metrics of strictly positive curvature on closed surfaces by

$$\boxed{N(g) \doteqdot \int_{M^2} \log(R\,\mathrm{Area})\, R d\mu,}$$

where $\mathrm{Area} \doteqdot \int_{M^2} d\mu$ is the area of g. We have the following entropy formula for solutions of the Ricci flow on closed surfaces with positive curvature.

EXERCISE 6.10. *Show that under the Ricci flow $\frac{\partial}{\partial t}g = -2\operatorname{Rc} = -Rg$ on surfaces with positive curvature, we have*

$$\frac{dN}{dt} = -\int_{M^2} \frac{|\nabla R|^2}{R} d\mu + \int_{M^2} (R-r)^2 d\mu \tag{6.3}$$

$$= -\int_{M^2} |\nabla \log R - \nabla f|^2 R d\mu - 2\int_{M^2} \left|\nabla\nabla f - \frac{1}{2}\Delta f \cdot g\right|^2 d\mu \leq 0 \tag{6.4}$$

where f is defined by $\Delta f = r - R$.

Equation (6.4) is related to a Poincaré type inequality, which holds in all dimensions on closed manifolds with positive Ricci curvature, due to Ben Andrews.

THEOREM 6.11 (Andrews' Poincaré type inequality). *Let (M^n, g) be a closed Riemannian manifold with positive Ricci curvature. If φ is any function with $\int_{M^n} \varphi d\mu = 0$ and if F_{ij} is any trace-free symmetric 2-tensor, then*

$$\frac{n}{n-1}\int_{M^n} \varphi^2 d\mu \leq \int_{M^n} |F_{ij}|^2 d\mu + \int_{M^n} \left(\operatorname{Rc}^{-1}\right)^{ij} (\nabla_i\varphi - \nabla_k F_{ki})(\nabla_j \varphi - \nabla_\ell F_{\ell j})\, d\mu. \tag{6.5}$$

COROLLARY 6.12. *If* $\int_{M^n} \varphi d\mu = 0$ *and* (M^n, g) *is closed with positive Ricci curvature, then*

$$\frac{n}{n-1} \int_{M^n} \varphi^2 d\mu \leq \int_{M^n} \left(\mathrm{Rc}^{-1}\right)^{ij} \nabla_i \varphi \nabla_j \varphi d\mu.$$

Note that Hamilton's inequality (6.3):

$$\int_{M^2} (R-r)^2 \, d\mu \leq \int_{M^2} \frac{|\nabla R|^2}{R} \, d\mu \tag{6.6}$$

is the special case of (6.5) where $n = 2$, $F_{ij} = 0$ and $\varphi = R - r$.

PROOF. Since $\int_{M^n} \varphi d\mu = 0$, there exists a function β on M^n such that $\Delta \beta = \varphi$. Starting from (**??**) we have

$$\int_{M^n} \left| \nabla_i \nabla_j \beta - \frac{\Delta \beta}{n} g_{ij} \right|^2 d\mu = \frac{n-1}{n} \int_{M^n} \varphi^2 d\mu - \int_{M^n} R_{ij} \nabla_i \beta \nabla_j \beta d\mu.$$

For any $a \in \mathbb{R}$ (the optimal choice for a will be determined from the computations below) we have

$$\begin{aligned} \int_{M^n} & \left| -a F_{ij} + \nabla_i \nabla_j \beta - \frac{\Delta \beta}{n} g_{ij} \right|^2 d\mu \\ &= \int_{M^n} \left(a^2 \left| F_{ij} \right|^2 d\mu - 2a F_{ij} \nabla_i \nabla_j \beta + \frac{n-1}{n} \varphi^2 - R_{ij} \nabla_i \beta \nabla_j \beta \right) d\mu. \end{aligned} \tag{6.7}$$

For any $b \in \mathbb{R}$ we also compute

$$\begin{aligned} & \int_{M^n} \left(\mathrm{Rc}^{-1}\right)^{ij} \left(\nabla_i \varphi - \nabla_k F_{ki} + b R_{ik} \nabla_k \beta \right) \left(\nabla_j \varphi - \nabla_\ell F_{\ell j} + b R_{j\ell} \nabla_\ell \beta \right) d\mu \\ & = \int_{M^n} \left(\mathrm{Rc}^{-1}\right)^{ij} \left(\nabla_i \varphi - \nabla_k F_{ki} \right) \left(\nabla_j \varphi - \nabla_\ell F_{\ell j} \right) d\mu \\ & + \int_{M^n} \left(b^2 R_{k\ell} \nabla_k \beta \nabla_\ell \beta - 2b \varphi^2 + 2b \nabla_k \nabla_i \beta F_{ki} \right) d\mu, \end{aligned} \tag{6.8}$$

where we integrated by parts and used the definition of β. Taking $b = 1/a$ and adding a^2 times equation (6.8) to (6.7), we obtain

$$\begin{aligned} 0 & \leq \int_{M^n} \left| -a F_{ij} + \nabla_i \nabla_j \beta - \frac{\Delta \beta}{n} g_{ij} \right|^2 d\mu \\ & + a^2 \int_{M^n} \left(\mathrm{Rc}^{-1}\right)^{ij} \left(\nabla_i \varphi - \nabla_k F_{ki} + b R_{ik} \nabla_k \beta \right) \left(\nabla_j \varphi - \nabla_\ell F_{\ell j} + b R_{j\ell} \nabla_\ell \beta \right) d\mu \\ & = \int_{M^n} \left(a^2 \left| F_{ij} \right|^2 + \left(\frac{n-1}{n} - 2a \right) \varphi^2 \right) d\mu \\ & + a^2 \int_{M^n} \left(\mathrm{Rc}^{-1}\right)^{ij} \left(\nabla_i \varphi - \nabla_k F_{ki} \right) \left(\nabla_j \varphi - \nabla_\ell F_{\ell j} \right) d\mu. \end{aligned}$$

Hence

$$\left(\frac{2}{a}-\frac{n-1}{na^2}\right)\int_{M^n}\varphi^2 d\mu \le \int_{M^n}|F_{ij}|^2 d\mu + \int_{M^n}\left(\mathrm{Rc}^{-1}\right)^{ij}\left(\nabla_i\varphi-\nabla_k F_{ki}\right)\left(\nabla_j\varphi-\nabla_\ell F_{\ell j}\right)d\mu.$$

Since $\frac{2}{a}-\frac{n-1}{na^2}\le\frac{n}{n-1}$ with equality when $a=\frac{n-1}{n}$, it is best to take $a=\frac{n-1}{n}$, in which case we obtain (6.5). □

An interesting special case of (6.5) is obtained by taking $\varphi = R-r$ and $F_{ij}=\alpha\left(R_{ij}-\frac{1}{n}Rg_{ij}\right)$, where $\alpha\in\mathbb{R}$. Here we have

$$\frac{n}{n-1}\int_{M^n}(R-r)^2 d\mu \le \alpha^2\int_{M^n}\left|R_{ij}-\frac{1}{n}Rg_{ij}\right|^2 d\mu + \left(1-\alpha\left(\frac{1}{2}-\frac{1}{n}\right)\right)^2\int_{M^n}\left(\mathrm{Rc}^{-1}\right)^{ij}\nabla_i R\nabla_j R d\mu.$$

When $n=2$, this is equivalent to (6.6). When $n\ge 3$, of particular note is when $\alpha=\left(\frac{1}{2}-\frac{1}{n}\right)^{-1}$.

Corollary 6.13. *If $n\ge 3$, then*

$$\int_{M^n}(R-r)^2 d\mu \le \frac{4n(n-1)}{(n-2)^2}\int_{M^n}\left|R_{ij}-\frac{1}{n}Rg_{ij}\right|^2 d\mu. \tag{6.9}$$

Note that a direct consequence of (6.9) is that if $R_{ij}\equiv\frac{1}{n}Rg_{ij}$, then $R\equiv r$. This is a fact which we originally derived from the contracted second Bianchi identity.

Problem 6.14 (Andrews' inequality applications). *For dimensions $n\ge 3$, is there an entropy estimate for solutions of the Ricci flow with some sort of positive curvature which follows from Andrews' Poincaré inequality?*

3. The gradient of Hamilton's entropy is the matrix Harnack

Consider the **space of smooth metrics** $\mathfrak{Met}$ on M^n. $\mathfrak{Met}$ is an open cone in the infinite dimensional vector space of symmetric 2-tensors. Its tangent space $T_g\mathfrak{Met}$ at any metric g is naturally isomorphic to the space of symmetric 2-tensors $C^\infty\left(T^*M^n\otimes_S T^*M^n\right)$. We may think of $\mathfrak{Met}$ as an infinite dimensional Riemannian manifold with the L^2**-metric**, defined by

$$\left\langle a_{ij},b_{ij}\right\rangle_{L^2}(g)\doteqdot\int_{M^n}g^{ik}g^{j\ell}a_{ij}b_{k\ell}d\mu_g, \tag{6.10}$$

for $g\in\mathfrak{Met}$ and $a_{ij},b_{ij}\in C^\infty\left(T^*M^n\otimes_S T^*M^n\right)\cong T_g\mathfrak{Met}$. If $F:\mathfrak{Met}\to\mathbb{R}$ is a smooth functional, then its **gradient** $\nabla F(g)$ at g is the symmetric 2-tensor defined by

$$\left\langle\nabla F(g)_{ij},h_{ij}\right\rangle_{L^2}(g)\doteqdot D_h F(g)\doteqdot\lim_{t\to 0}\frac{F(g+th)-F(g)}{t}$$

for all $h \in T_g\mathfrak{Met}$, where $D_h F(g)$ is called the **Frechet derivative** of F at g in the direction h.

Recall that the (unnormalized) entropy of a surface with positive curvature is

$$E(g) \doteqdot \int_{M^2} \log R \cdot R d\mu,$$

where R is the scalar curvature. We compute the **gradient of the entropy** in the space of metrics. Let $h_{ij} \doteqdot \frac{\partial}{\partial s} g_{ij}$ be a variation of g_{ij}. The variation of R (in all dimensions) is

$$\frac{\partial R}{\partial s} = -\Delta H + \nabla_p \nabla_q h_{pq} - h \cdot \mathrm{Rc}$$

and the variation of the area form is $\frac{\partial}{\partial s} d\mu = \frac{1}{2} H d\mu$. Since $\mathrm{Rc} = \frac{1}{2} R g$ when $n = 2$, we have after an integration by parts

$$\frac{d}{ds} E(g) = \int_{M^2} h_{pq} \left(-\Delta \log R \cdot g_{pq} + \nabla_p \nabla_q \log R - \frac{1}{2} R g_{pq} \right) d\mu.$$

That is

LEMMA 6.15 (Gradient of entropy is Harnack).

$$\begin{aligned} -\nabla E(g_{ij}) &= \Delta \log R \cdot g - \nabla\nabla \log R + \frac{1}{2} R g \\ &= J\left(\nabla\nabla \log R + \frac{1}{2} R g \right), \end{aligned}$$

where the operator J is defined below in (8.51).

The quantity $\nabla\nabla \log R + \frac{1}{2} R g$ is the matrix Harnack quantity which appears in (8.24).

PROBLEM 6.16. *Is H.-D. Cao's matrix Harnack quadratic (see* [**72**]*) related to some entropy for closed Kähler manifolds? Here we expect to need to make some positive curvature assumption, such as positive bisectional curvature.*

4. The Yamabe flow and Aleksandrov reflection

For the **Yamabe flow**

$$\frac{\partial}{\partial t} g = -R g \tag{6.11}$$

on locally conformally flat manifolds, there is a *gradient estimate for the conformal factor* by Rugang Ye [**533**] (see Bartz-Struwe-Ye [**40**] for dimension 2). Recall that a Riemannian manifold is said to be **locally conformally flat** if about point there exists a local coordinate system in which $g_{ij} = v\delta_{ij}$ for some positive function v. For solutions to the Yamabe problem ($R \equiv r$), which is the elliptic analogue of the Yamabe flow, Schoen [**442**] proved a related 'no bubbling' result using the **Aleksandrov reflection method** along the lines of the work of Gidas-Ni-Nirenberg [**224**] and the positive mass theorem to embed the universal cover into S^n [**446**]. Ye's gradient estimate also applies to any weakly parabolic conformal flow of metrics on S^n, which we shall now describe (see [**135**]).

Let g_{S^n} denote the standard metric on S^n, which we shall think of as the unit sphere in $\mathbb{R}^{n+1}$. We consider a general class of conformal flows of metrics conformal to g_{S^n}. In particular, given $g_0 = e^{u_0} g_{S^n}$, suppose that $g(t) = e^{u(t)} g_{S^n}$ is a solution to

$$
\begin{aligned}
\frac{\partial g}{\partial t}(x,t) &= F\left(\left(\mathrm{Rc} - \frac{R}{2(n-1)} g\right)(x,t), \frac{g(x,t)}{g_{S^n}}, t\right) \cdot g(x,t) \\
g(x,0) &= g_0(x)
\end{aligned} \tag{6.12}
$$

for $x \in S^n$, $t \in [0,T)$, where $F : S^2 T^* S^n \times \mathbb{R}^+ \times [0,T) \to \mathbb{R}$.

Recall that if $g = e^u g_{S^n}$, then

$$
R = (n-1) e^{-u} \left(n - \Delta u - \frac{n-2}{4} |\nabla u|^2\right),
$$

where Δ and $|\cdot|$ are with respect to g_{S^n}. Hence the Yamabe flow $\frac{\partial}{\partial t} g = -Rg$ is equivalent to

$$
\frac{\partial u}{\partial t} = (n-1) e^{-u} \left(\Delta u + \frac{n-2}{4} |\nabla u|^2 - n\right). \tag{6.13}
$$

EXERCISE 6.17. *If* $\frac{\partial}{\partial s} g_{ij} = f g_{ij}$, *then*

$$
\frac{\partial}{\partial s}\left(R_{ij} - \frac{1}{2(n-1)} R g_{ij}\right) = -\frac{n-2}{2} \nabla_i \nabla_j f. \tag{6.14}
$$

Let $D_1 F$, $D_2 F$, $D_3 F$ denote the partial derivatives of F. Using (6.14) show that if $\frac{\partial}{\partial s} g_{ij} = f g_{ij}$, then the **linearization** of F is given by

$$
\frac{\partial}{\partial s}\left(F\left(\mathrm{Rc} - \frac{R}{2(n-1)} g, \frac{g}{g_{S^n}}, t\right)\right) = D_1 F\left(-\frac{n-2}{2} \nabla_i \nabla_j f\right) + D_2 F\left(f \frac{g}{g_{S^n}}\right).
$$

From this we see that if F is C^1 in the first component and $D_1 F \leq 0$, then (6.12) is weakly parabolic.

Of particular interest is the **Yamabe flow**:

$$
\frac{\partial g}{\partial t} = -Rg,
$$

and the corresponding **normalized Yamabe flow**

$$
\frac{\partial g}{\partial t} = (r - R) g
$$

where r is the average scalar curvature. We make the following monotonicity assumption on F which ensures the weak parabolicity of the evolution equation (6.12). We shall always assume F is Lipschitz continuous in the first component.

(1) If $n \geq 3$, we assume

$$F(\beta_1, h, t) \geq F(\beta_2, h, t) \tag{6.15}$$

for all $\beta_1, \beta_2 \in S^2T^*S^n$ with $\beta_1 \leq \beta_2$, $h > 0$ and $t \in [0, T)$.

(2) If $n = 2$, we assume the evolution equation is of the form

$$\frac{\partial g}{\partial t}(x,t) = F(R(x,t), t) \cdot g(x,t)$$

where

$$F(\rho, t) \text{ is a nonincreasing function of } \rho. \tag{6.16}$$

We have the following gradient estimate, proved by [**533**] for the Yamabe flow.

THEOREM 6.18 (Gradient estimate for weakly conformal flows). *If $g(t) \doteqdot e^{u(t)} g_{S^n}$, $t \in [0,T)$, is a solution to (6.12) under the hypothesis (6.15) when $n \geq 3$ or the hypothesis (6.16) when $n = 2$, then there exists a constant $C < \infty$ depending only on $g(0)$ such that*

$$\boxed{|\nabla u(x,t)| \leq C} \quad \textit{for all } x \in S^n \textit{ and } t \in [0,T).$$

From integrating this estimate along great circles in S^n, we have:

COROLLARY 6.19 (Classical type Harnack estimate).

$$\max_{S^n} u(t) \leq \min_{S^n} u(t) + C.$$

This is a Harnack estimate since we may write it in terms of the conformal factor as:

$$\boxed{\max_{S^n} e^{u(t)} \leq e^C \min_{S^n} e^{u(t)}.} \tag{6.17}$$

To obtain uniform upper and lower bounds for u, we need a normalizing condition. For example, if the volume $\operatorname{Vol}(g(t)) = \int_{M^n} e^{\frac{n}{2}u(t)} d\mu_{S^n}$ is bounded from above and below by positive constants, then $|u| \leq C$ for some $C < \infty$. We now can derive the following result of Ye [**533**].

THEOREM 6.20 (Convergence of the Yamabe flow of LCF manifolds). *If (M^n, g_0) is a closed, locally conformally flat manifold with positive Yamabe invariant:*

$$\inf_{g = e^u g_0} \operatorname{Vol}(g)^{-(n-2)/n} \int_{M^n} R_g d\mu_g > 0,$$

then there exists a solution to the Yamabe flow $g(t)$, defined for all $t \in [0,\infty)$ and with $g(0) = g_0$, such that $g(t)$ converges in each C^k-norm to a C^∞ metric g_∞ conformal to g_0 with constant scalar curvature $R(g_\infty) \equiv r$.

PROOF. Since the Yamabe flow (6.13) is strictly parabolic, there exists a solution for a short time interval $[0, \varepsilon)$. Let $[0, T)$ be the maximal time interval of existence. We claim $T = \infty$. By the Harnack estimate (6.17), the Yamabe flow in terms of the conformal factor (6.13) is a second order uniformly parabolic equation. Hence by the estimate of Krylov-Safonov [**333**], u satisfies a uniform Hölder estimate. By parabolic Schauder theory (which is similar to elliptic Schauder theory), u satisfies a $C^{2k,\alpha}$ estimate for all $k \geq 1$ (see [**349**] for example). Hence, unless $T = \infty$, we can continue the solution past time T. Next we may apply L. Simon's asymptotic theorem to conclude that $g(t)$ converge in each C^k norm to a C^∞ metric g_∞. From (2.4) we see that the scalar curvature evolves by:

$$\frac{\partial R}{\partial t} = (n-1)\,\Delta R + R^2 - rR,$$

so that

$$\frac{dr}{dt} = \frac{d}{dt}\left(\frac{\int_M R d\mu}{\int_M d\mu}\right) = -\frac{n-2}{2}\frac{\int_M (R-r)^2 d\mu}{\int_M d\mu}.$$

Therefore $R(g_\infty) \equiv r$. □

Given a point $N \in S^n$, let

$$\sigma_N : S^n - \{N\} \to N^\perp$$

be the **stereographic projection map** defined by

$$\sigma_N(y) \doteqdot \frac{y - \langle y, N\rangle N}{1 - \langle y, N\rangle}.$$

Here $N^\perp$ is the hyperplane $N^\perp \doteqdot \left\{x \in \mathbb{R}^{n+1} : \langle x, N\rangle = 0\right\}$. The inverse of the stereographic projection map is given by

$$\sigma_N^{-1} : N^\perp \to S^n - \{N\}$$

where

$$\sigma_N^{-1}(x) \doteqdot \frac{2x + \left(|x|^2 - 1\right) N}{|x|^2 + 1}$$

for $x \in N^\perp$. We think of N as the **north pole**. Accordingly we define the **south pole** by $S \doteqdot -N$.

We are interested in the following conformal diffeomorphisms of S^n :

(1) (**Reflection about the equator**)

$$\rho_N : S^n \to S^n$$

where

$$\rho_N(y) \doteqdot y - 2\langle y, N\rangle N.$$

(2) (**Conformal dilations**)

$$\varphi_N^\alpha \doteqdot \sigma_N^{-1} \circ \alpha \circ \sigma_N : S^n \to S^n$$

where $\alpha : N^\perp \to N^\perp$ is multiplication by $\alpha \in (0, \infty)$.

The conformal dilations φ_N^α fix the poles N and S, and $\varphi_N^1 = \mathrm{Id}_{S^n}$. The direction of the dilation (or stretch) is given as follows: for $\alpha > 1$, φ_N^α takes points closer to N, while for $\alpha < 1$, φ_N^α takes points closer to S. In terms of pull backs this means the following: for any metric g on S^n, for $\alpha > 1$, $(\varphi_N^\alpha)^* g$ is concentrated more near S (for $\alpha < 1$, near N.)

The idea of the Aleksandrov reflection method begins with the following. Let $H_N^+ \doteqdot \{y \in S^n : \langle y, N\rangle \geq 0\}$ denote the **northern hemisphere** (and similarly define the **southern hemisphere** H_N^-.)

LEMMA 6.21. *For any metric $g = e^u g_{S^n}$, there exists a constant $\alpha_0 \in (0,1)$ such that for all $\alpha \in (0, \alpha_0]$ and $N \in S^n$, we have*

$$(\varphi_N^\alpha)^* g \geq \rho_N^* (\varphi_N^\alpha)^* g \quad \text{on } H_N^+.$$

PROOF. Intuitively, this is quite obvious, so we skip the proof. For an elementary and detailed proof, we refer the reader to Proposition 3.1 in [**135**]. □

Define $u_N^\alpha : S^n \to \mathbb{R}$ by $(\varphi_N^\alpha)^* g \doteqdot \exp(u_N^\alpha) g_{S^n}$. Since

$$\rho_N^* (\varphi_N^\alpha)^* g = \exp(u_N^\alpha \circ \rho_N) g_{S^n},$$

the lemma is equivalent to:

COROLLARY 6.22. *For every $N \in S^n$ and every $\alpha \in (0, \alpha_0]$, we have*

$$u_N^\alpha \geq u_N^\alpha \circ \rho_N \quad \text{on } H_N^+. \tag{6.18}$$

From this we can see that the constant $\alpha_0 \in (0,1)$ determines the following **gradient estimate** for the conformal factor u on S^n. This proposition will enable the Aleksandrov reflection method to provide an estimate for $|\nabla u|$ for weakly parabolic conformal flows on S^n.

PROPOSITION 6.23 (Aleksandrov reflection implies gradient estimate). *For any metric $g = e^u g_{S^n}$, if $\alpha_0 \in (0,1)$ is as in the above lemma, then*

$$|\nabla u(x)| \leq \frac{1 - \alpha_0^2}{\alpha_0} \tag{6.19}$$

for all $x \in S^n$.

PROOF. (*Sketch, see* [**135**] *for details.*) Let

$$E_N \doteqdot \partial H_N^+ = \{y \in S^n : \langle y, N\rangle = 0\}$$

denote the **equator**. First we observe that (6.18) implies for all $\alpha \in (0, \alpha_0]$ and $N \in S^n$

$$\langle \nabla u_N^\alpha (y), N\rangle \geq 0$$

for all $y \in E_N$. The vector field N is tangent to S^n on the equator E_N. We extend $N|_{E_N}$ to a vector field $\bar{N}$ defined on $S^n - \{N, S\}$ by parallel translating N along all of the great circles which pass through N and S. Since

$$\langle \nabla u_N^\alpha (y), N\rangle = \frac{2\alpha}{\alpha^2 + 1} \left\langle \nabla u(\varphi_N^\alpha (y)), \bar{N}\right\rangle + \frac{2\left(1 - \alpha^2\right)}{\alpha^2 + 1},$$

in terms of the original conformal factor, this implies

$$\left\langle \nabla u \left(\varphi_N^{\alpha_0} \left(y \right) \right), \bar{N} \right\rangle \geq -\frac{1-\alpha_0^2}{\alpha_0}. \tag{6.20}$$

for all $y, N \in S^n$ such that $\langle y, N \rangle = 0$. Now we want to see why this implies the gradient estimate (6.19). Given any point $x \in S^n$, define

$$J_{\alpha_0}\left(x\right) \doteqdot \left\{ y \in S^n : \varphi_N^{\alpha_0}\left(y\right) = x \text{ for some } N \in S^n \text{ with } \left\langle y, N \right\rangle = 0 \right\}.$$

It is not hard to see that

$$J_{\alpha_0}\left(x\right) = \left\{ y \in S^n : \left\langle y, x \right\rangle = \frac{2\alpha_0}{\alpha_0^2 + 1} \right\}$$

is an $(n-1)$-sphere centered at x. Let

$$K_{\alpha_0} \doteqdot \left\{ N \in S^n : \varphi_N^{\alpha_0}\left(y\right) = x \text{ for some } y \in S^n \text{ with } \left\langle y, N \right\rangle = 0 \right\}.$$

Then

$$K_{\alpha_0}\left(x\right) = \left\{ N \in S^n : \left\langle N, x \right\rangle = \frac{\alpha_0^2 - 1}{\alpha_0^2 + 1} \right\}$$

which is also an $(n-1)$-sphere centered at x. From this and (6.20) it is not hard to see that for any $x \in S^n$

$$\left\langle \nabla u\left(x\right), \bar{N} \right\rangle \geq -\frac{1-\alpha_0^2}{\alpha_0}$$

for all $\bar{N} \in S^n$ such that $\left\langle \bar{N}, x \right\rangle = 0$. This completes the proof of (6.19). □

Now we are ready to give the

Proof of Theorem 6.18. Remarkably, the proof is quite easy now. By Lemma 6.21, there exists $\alpha_0 \in (0,1)$ such that for all $\alpha \in (0, \alpha_0]$ and $N \in S^n$

$$\left(\varphi_N^{\alpha}\right)^* g\left(0\right) \geq \rho_N^* \left(\varphi_N^{\alpha}\right)^* g\left(0\right) \quad \text{on } H_N^+. \tag{6.21}$$

By the conformal invariance of equation (6.12), both $\left(\varphi_N^{\alpha}\right)^* g\left(t\right)$ and $\rho_N^* \left(\varphi_N^{\alpha}\right)^* g\left(t\right)$ are solutions to the same weakly parabolic flow (6.12). By the maximum principle for scalars and (6.21), we have

$$\left(\varphi_N^{\alpha}\right)^* g\left(t\right) \geq \rho_N^* \left(\varphi_N^{\alpha}\right)^* g\left(t\right) \quad \text{on } H_N^+ \text{ for all } t \in [0,T) \text{ and } \alpha \in (0, \alpha_0].$$

By the proof of Proposition 6.23, we conclude that

$$\left|\nabla u\left(x,t\right)\right| \leq \frac{1-\alpha_0^2}{\alpha_0} \quad \text{for all } x \in S^n \text{ and } t \in [0,T).$$

□

4.1. L. Simon's asymptotic theorem. Let (M^n, g) be a closed Riemannian manifold and consider an energy functional of the form:

$$\boxed{\mathcal{E}(u) = \int_{M^n} E(\nabla u, u, x)\, d\mu,}$$

defined for $u \in C^\infty(M^n)$, where

(1) $E : TM^n \times \mathbb{R} \times M^n \to \mathbb{R}$ depends smoothly on ∇u, u and x,
(2) $E(p, z, x)$ is **uniformly convex** in p for $|p| + |z|$ sufficiently small, so that
$$\left.\frac{d^2}{ds^2} E(sq, 0, x)\right|_{s=0} \geq c\,|q|^2$$
where $c > 0$ is independent of $x \in M^n$ and $q \in T_x M^n$,
(3) $E(p, z, x)$ depends **analytically** on p and z, **uniformly** in $x \in M^n$, for $|p| + |z|$ sufficiently small, so that there exists $\varepsilon > 0$ such that
$$E(p + s_1 q, z + s_2 w, x) = \sum_{m_1, m_2 \geq 0} E_{m_1, m_2}(p, q, z, w, x)\, s_1^{m_1} s_2^{m_2}$$
and
$$\left| \sum_{m_1 + m_2 = k} E_{m_1, m_2}(p, q, z, w, x)\, s_1^{m_1} s_2^{m_2} \right| \leq 1$$
for $|p|, |q|, |z|, |w| < \varepsilon$, $s_1^2 + s_2^2 < 1$ and $k \geq 1$.
(4) The gradient (or Euler-Lagrange operator) of $\mathcal{E}$, defined by
$$\langle \operatorname{grad} \mathcal{E}(u), v \rangle_{L^2(M^n)} = \left.\frac{d}{ds} \mathcal{E}(u + sv)\right|_{s=0},$$
satisfies $\operatorname{grad} \mathcal{E}(0) = 0$.

Consider the second-order parabolic partial differential equation:

$$\frac{\partial u}{\partial t} = -\operatorname{grad} \mathcal{E}(u).$$

Assume that

$$\mathcal{E}(u(t)) \geq \mathcal{E}(0) - \delta$$

for δ sufficiently small implies that $u(t) \to u_\infty$, where u_∞ is a solution to

$$\operatorname{grad} \mathcal{E}(u_\infty) = 0.$$

The energy functional corresponding to the Yamabe flow is

$$\begin{aligned}
\mathcal{Y}(g) &\doteqdot \int_{M^n} R\, d\mu \\
&= (n-1) \int_{M^n} e^{-u} \left(n - \Delta u - \frac{n-2}{4} |\nabla u|^2 \right) e^{\frac{n}{2} u}\, d\mu_{S^n} \\
&= (n-1) \int_{M^n} \left(n + \frac{n-2}{4} |\nabla u|^2 \right) e^{\frac{n-2}{2} u}\, d\mu_{S^n}.
\end{aligned}$$

Here

$$E(p,z,x) = \left(n + \frac{n-2}{4}|p|^2\right) e^{\frac{n-2}{2}z}.$$

In particular

$$\left.\frac{d^2}{ds^2} E(sq,0,x)\right|_{s=0} = \frac{n-2}{2}|q|^2.$$

and

$$E(p+s_1 q, z+s_2 w, x) = \left(n + \frac{n-2}{4}|p+s_1 q|^2\right) e^{\frac{n-2}{2}z+s_2 w}.$$

5. Isoperimetric estimate

5.1. Evolution of the length of a geodesic. Let $(M^n, g(t))$ be a solution to the Ricci flow, $\gamma : [a,b] \to M^n$ be a fixed path, and $S \doteqdot \frac{d\gamma}{ds}$. Given a time t_0, assume that γ is parametrized by arc length with respect to $g(t_0)$ so that $|S| = 1$ at $t = t_0$. The evolution of the length of γ with respect to $g(t)$ is given by

$$\begin{aligned} \left.\frac{d}{dt}\right|_{t=t_0} \mathrm{L}_{g(t)}(\gamma) &= \int_a^b \left(\left.\frac{\partial}{\partial t}\right|_{t=t_0} \sqrt{g(t)(S,S)}\right) ds \\ &= -\int_a^b \mathrm{Rc}_{g(t_0)}(S,S)\, ds. \end{aligned} \tag{6.22}$$

Further assume that γ_0 is a geodesic with respect to $g(t_0)$. Recall the second variation formula (1.120):

$$\begin{aligned} \left.\frac{\partial^2}{\partial r^2}\right|_{r=0} \mathrm{L}_{g(t_0)}(\gamma_r) &= \int_a^b \left(|\nabla_S R|^2 - \langle \nabla_S R, S\rangle^2 - \langle \mathrm{Rm}(R,S)S, R\rangle\right) ds \\ &\doteqdot I(R,R). \end{aligned}$$

where the norms and inner products are with respect to $g(t_0)$. Now let $\{e_i\}_{i=1}^{n-1}$ be an orthonormal basis of parallel vector fields spanning $S^\perp$ along γ_0. Since $\nabla_S e_i = 0$ along γ_0, we have

$$I(e_i,e_i) = -\int_a^b \langle \mathrm{Rm}(e_i,S)S, e_i\rangle\, ds.$$

Hence we have

LEMMA 6.24 (Heat type equation for lengths of geodesics).

$$\left.\frac{d}{dt}\right|_{t=t_0} \mathrm{L}_{g(t)}(\gamma) = \sum_{i=1}^{n-1} I(e_i,e_i) = \sum_{i=1}^{n-1} \left.\frac{\partial^2}{\partial r_i^2}\right|_{r_i=0} \mathrm{L}_{g(t_0)}(\gamma_{r_i})$$

where γ_{r_i} is a 1-parameter family of paths with variation vector field e_i at $r_i = 0$.

So in essence, the length of a geodesic satisfies a heat-like equation.

5.2. Isoperimetric estimate for surfaces. In this section we present Hamilton's monotonicity formula for the isoperimetric constant for solutions of the Ricci flow on the 2-sphere. Throughout this section we give statements of facts and results without proof; for the detailed proofs see [**266**] and [**153**] §5.14, p. 156ff; We first recall some basic facts about the isoperimetric constant; see also Volume 2 and the references therein.

Let $\left(M^2, g\right)$ be a Riemannian surface diffeomorphic to the 2-sphere. Given an embedded closed curve γ separating M^2 into two open surfaces M_+^2 and M_-^2, we define the **isoperimetric ratio** of γ by

$$C_S(\gamma) \doteqdot \mathrm{L}(\gamma)^2 \left(\frac{1}{\mathrm{A}\left(M_+^2\right)} + \frac{1}{\mathrm{A}\left(M_-^2\right)} \right) = \mathrm{L}(\gamma)^2 \frac{\mathrm{A}\left(M^2\right)}{\mathrm{A}\left(M_+^2\right) \cdot \mathrm{A}\left(M_-^2\right)}, \tag{6.23}$$

where L and A denote the length and area. The **isoperimetric constant** of $\left(M^2, g\right)$ is

$$C_S\left(M^2, g\right) \doteqdot \inf_\gamma C_S(\gamma),$$

where the infimum is taken over all smooth embedded (not necessarily connected) γ that separate M^2. It can be shown that there exists a smooth embedded closed curve $\gamma_{\min}$ such that $C_S(\gamma_{\min}) = C_S\left(M^2, g\right)$. Similar to C_S we define

$$C_H\left(M^2, g\right) \doteqdot \inf_\gamma C_H(\gamma),$$

where the infimum is taken over all smooth embedded (connected) **loops** γ. It is not hard to see

$$C_S\left(M^2, g\right) \leq C_H\left(M^2, g\right) \leq 4\pi.$$

Furthermore, if $C_H\left(M^2, g\right) < 4\pi$, then there exists a smooth embedded loop $\gamma_{\min}$ such that $C_H(\gamma_{\min}) = C_H\left(M^2, g\right)$.

The main result of this section is the following.

THEOREM 6.25 (Monotonicity of the isoperimetric constant). *If $\left(M^2, g(t)\right)$ is a solution of the Ricci flow on a topological 2-sphere, then $C_H\left(M^2, g(t)\right)$ is monotonically nondecreasing for all t.*

The proof relies on deriving the following heat-type equation. Given an embedded loop γ_0 in M^2, by the Schoenflies Theorem, γ_0 separates M^2 into two disks M_+^2 and M_-^2. Let γ_ρ denote the **parallel curves** (of signed distance ρ) to γ_0, where $\gamma_\rho \subset M_+^2$ for $\rho > 0$, and which separate M^2 into two disks $M_+^2(\rho)$ and $M_-^2(\rho)$; we assume ρ is sufficiently small so that γ_ρ is smooth and embedded.

LEMMA 6.26 (Heat equation for isoperimetric ratios of parallel loops). *Let $\left(M^2, g(t)\right)$ be a solution of the Ricci flow on a topological 2-sphere and let γ_0 be an embedded loop. Given a time t_0, let γ_ρ denote the parallel loops with* *respect to $g(t_0)$. Then the isoperimetric ratios $C_H(\rho, t)$ of γ_ρ with respect to the metric $g(t)$ satisfy*

$$\boxed{\frac{\partial}{\partial t}(\log C_H) = \frac{\partial^2}{\partial \rho^2}(\log C_H) + \frac{\int_{\gamma_\rho} k\,ds}{L}\frac{\partial}{\partial \rho}(\log C_H) + \frac{4\pi - C_H}{A}\left(\frac{A_+}{A_-} + \frac{A_-}{A_+}\right)}$$

at $t = t_0$, where $A_\pm \doteqdot \mathrm{A}\left(M^2_\pm(\rho)\right)$, $A \doteqdot \mathrm{A}\left(M^2\right)$ and k is the geodesic curvature of γ_ρ.

Roughly speaking, Theorem 6.25 follows from Lemma 6.26 since if γ_0 is a minimizer of $C_H(\gamma, g(t_0))$ with $C_H(\gamma_0, g(t_0)) < 4\pi$, then

$$\left.\frac{\partial}{\partial t}\right|_{t=t_0}(\log C_H(\gamma_0; g(t))) \geq \frac{4\pi - C_H(\gamma_0; g(t_0))}{A}\left(\frac{A_+}{A_-} + \frac{A_-}{A_+}\right) > 0.$$

A simple geometric result establishes the following.

LEMMA 6.27 (Isoperimetric estimate implies injectivity radius estimate). *If $\left(M^2, g\right)$ is a topological 2-sphere, then*

$$\left[\mathrm{inj}\left(M^2, g\right)\right]^2 \geq \frac{\pi}{4K_{\max}} C_H\left(M^2, g\right)$$

where $K_{\max}$ is the maximum of the Gauss curvature.

So as a consequence, we have:

COROLLARY 6.28 (Injectivity radius estimate for the Ricci flow on S^2). *If $\left(M^2, g(t)\right)$ is a solution of the Ricci flow on a topological S^2, then*

$$\mathrm{inj}\left(M^2, g(t)\right) \geq \frac{c}{\sqrt{K_{\max}(t)}},$$

where $c \doteqdot \sqrt{\frac{\pi}{4} C_H\left(M^2, g(0)\right)} > 0$.

As a consequence of the isoperimetric estimate, we see that any singular solution of the Ricci flow on a topological S^2 is Type I; for if it were Type IIa, one would then obtain a cigar soliton as a limit, contradicting the isoperimetric estimate.

5.3. Isoperimetric estimate for Type I singular solutions in dimension three. Let $\left(M^3, g\right)$ be a closed Riemannian 3-manifold. Given $V \in (0, \mathrm{Vol}(g))$, let

$$G(V) \doteqdot \inf_{\Sigma^2} \mathrm{A}\left(\Sigma^2\right)$$

where the infimum is taken over smooth embedded surfaces Σ^2 which separate M^3 into two regions of volume V and $\mathrm{Vol}(g) - V$. Note that

$$\lim_{V \to 0} G(V) = \lim_{V \to \mathrm{Vol}(g)} G(V) = 0$$

and that G is an even function about $V = \mathrm{Vol}(g)/2$. The following result was proved in [**267**].

THEOREM 6.29 (Isoperimetric estimate for Type I singular solutions on 3-manifolds). *If* $\left(M^3, g(t)\right)$, $t \in [0,T)$, *is a Type I singular solution of the Ricci flow on a closed* 3*-manifold with* $\mathrm{Vol}(g(t)) \geq c(T-t)^{3/2}$, *then there exists a constant* $c > 0$ *such that if* $\Sigma^2 \subset M^3$ *is a smooth embedded surface separating* M^3 *into two regions with volumes at least* V, *then*

$$\mathrm{A}\left(\Sigma^2; g(t)\right) \geq c \min\left\{V^{2/3}, T-t\right\},$$

where $\mathrm{A}\left(\Sigma^2; g(t)\right)$ *denotes the area of* Σ^2 *with respect to* $g(t)$.

The proof of the theorem is a consequence of the following estimate.

LEMMA 6.30. *If* $\left(M^3, g(t)\right)$, $t \in [0,T)$, *satisfies the hypotheses of the above lemma and is normalized so that* $\min_{t=0} R \geq -\rho$, *then there exist constants* $A, B < \infty$ *such that for all* $V \in \left[0, \frac{1}{2}\mathrm{Vol}(g(t))\right]$ *and* $t \in [0,T)$

$$G(V,t) > e^{-\frac{2}{3}\rho t}\left(\frac{A}{V^{2/3}} + \frac{B}{T-t}\right)^{-1}.$$

where $G(V,t)$ *denotes* $G(V)$ *with respect to* $g(t)$.

Indeed, the lemma implies

$$\mathrm{A}\left(\Sigma^2; g(t)\right) > \frac{1}{2}e^{-\frac{2}{3}\rho T}\min\left\{A^{-1}V^{2/3}, B^{-1}(T-t)\right\}.$$

In turn, the isoperimetric estimate implies an injectivity radius estimate. This estimate has been superseded by Perelman's no local collapsing theorem, which not only provides an injectivity radius estimate for both Type I and IIa singular solutions, but also rules out local collapse and in particular the formation of the cigar soliton singularity model.

SKETCH OF PROOF OF LEMMA. We shall show that $G(V,t) > F(V,t)$ for all $V \in (0, \mathrm{Vol}(g(t)))$ and $t \in [0,T)$, where

$$F(V,t) \doteqdot e^{-\frac{2}{3}\rho t}\left(\frac{A}{V^{2/3}} + \frac{A}{(\mathrm{Vol}(g(t)) - V)^{2/3}} + \frac{B}{T-t}\right)^{-1}$$

and A and B are to be determined; F is an even function about $V = \mathrm{Vol}(g)/2$. We leave it to the reader to check that
1. for t sufficiently small and any V, $G(V,t) > F(V,t)$,
2. given t, for any V sufficiently small, $G(V,t) > F(V,t)$.
Thus if the lemma is false, then there exists $t_0 > 0$ such that there is a $V_0 \in (0, \frac{1}{2}\mathrm{Vol}(g(t_0))]$ where

$$G(V_0, t_0) = F(V_0, t_0)$$

for the first time. The idea is to obtain a contradiction by deriving heat-type equations for F and G and applying the maximum principle. There exists a smooth embedded surface Σ_0^2 of constant mean curvature such that

$$G(V_0, t_0) = \mathrm{A}\left(\Sigma_0^2; g(t_0)\right).$$

Let Σ_r^2 denote the parallel surfaces of signed distance r from Σ_0^2 with respect to $g(t_0)$, where Σ_r^2 is in the region of volume V_0 for $r<0$. For r sufficiently small, Σ_r^2 is embedded and diffeomorphic to Σ_0^2. Let

$$A(r,t) \doteqdot \mathrm{A}\left(\Sigma_r^2; g(t)\right)$$

and

$$V(r,t) \doteqdot \mathrm{Vol}\left(M_-^3(r); g(t)\right)$$

where $M_-^3(r)$ is the region enclosed by Σ_r^2 on the side which has

$$\mathrm{Vol}\left(M_-^3(0); g(t_0)\right) = V_0.$$

Recall

$$\frac{\partial A}{\partial t} = \frac{\partial^2 A}{\partial r^2} - 4\pi\chi\left(\Sigma_r^2\right).$$

On the other hand, using

$$\mathrm{A}\left(\Sigma_r^2; g(t_0)\right) \geq G\left(V(r,t_0), t_0\right) \geq F\left(V(r,t_0), t_0\right)$$

for all r sufficiently small, and $\mathrm{A}\left(\Sigma_0^2; g(t_0)\right) = F\left(V(0,t_0), t_0\right)$, it follows that at $(0,t_0)$

$$\frac{\partial A}{\partial r} = \frac{\partial F}{\partial V}\frac{\partial V}{\partial r}$$

and

$$\frac{\partial^2 A}{\partial r^2} \geq \left(\frac{\partial V}{\partial r}\right)^2 \frac{\partial^2 F}{\partial V^2} + \frac{\partial^2 V}{\partial r^2}\frac{\partial F}{\partial V}.$$

Similarly, from $\mathrm{A}\left(\Sigma_0^2; g(t)\right) \geq F\left(V(0,t), t\right)$ for $t \leq t_0$, we have at $(0,t_0)$

$$\frac{\partial A}{\partial t} \leq \frac{\partial F}{\partial t} + \frac{\partial V}{\partial t}\frac{\partial F}{\partial V}.$$

One can show $\chi\left(\Sigma_r^2\right) \leq 0$ (this requires the Gauss-Bonnet formula, Gauss equation, Type I condition to estimate the curvature, see [**267**], §23 for details). Hence

$$\frac{\partial A}{\partial t} \geq \frac{\partial^2 A}{\partial r^2}$$

which implies

$$\frac{\partial F}{\partial t} + \frac{\partial V}{\partial t}\frac{\partial F}{\partial V} \geq \left(\frac{\partial V}{\partial r}\right)^2 \frac{\partial^2 F}{\partial V^2} + \frac{\partial^2 V}{\partial r^2}\frac{\partial F}{\partial V}.$$

Using $\frac{\partial V}{\partial t} \leq \rho V$ (since $R \geq -\rho$) and $\frac{\partial V}{\partial r} = F$, $\frac{\partial^2 V}{\partial r^2} = F\frac{\partial F}{\partial V}$ (again see [**267**], §23 for details), we obtain at $(0,t_0)$

$$\frac{\partial F}{\partial t} + \rho V_0 \frac{\partial F}{\partial V} \geq F^2 \frac{\partial^2 F}{\partial V^2} + F\left(\frac{\partial F}{\partial V}\right)^2$$

where we have assumed $0 < V_0 \leq \frac{1}{2}\mathrm{Vol}\left(g(t_0)\right)$. The contradiction is obtained by showing that for A and B sufficiently large, by using the definition of F the opposite inequality holds. $\square$

Let $\left(M^3, g\right)$ be a closed Riemannian 3-manifold. The **isoperimetric constant** $C_I = C_I\left(M, g\right)$ is defined by

$$C_I \doteqdot \inf_{\Sigma} \frac{\operatorname{Area}\left(\Sigma\right)}{\min\left\{\operatorname{Vol}\left(M_1\right)^{2/3}, \operatorname{Vol}\left(M_2\right)^{2/3}\right\}},$$

where the infimum is taken over all surfaces Σ^2 (not necessarily connected) which divide M^3 into two regions M_1 and M_2. Hence for any surface Σ dividing M^3 into two regions M_1 and M_2 we have

$$\operatorname{Area}\left(\Sigma\right) \geq C_I \cdot \min\left\{\operatorname{Vol}\left(M_1\right)^{2/3}, \operatorname{Vol}\left(M_2\right)^{2/3}\right\}.$$

Now we estimate the volume of small enough balls. For any point $p \in M^3$, we compute

$$\begin{aligned}\frac{d}{dr}\operatorname{Vol}\left(B\left(p,r\right)\right) &= \operatorname{Area}\left(\partial B\left(p,r\right)\right) \\ &\geq C_I \cdot \min\left\{\operatorname{Vol}\left(B\left(p,r\right)\right)^{2/3}, \operatorname{Vol}\left(M^3 - B\left(p,r\right)\right)^{2/3}\right\}.\end{aligned}$$

Given a constant $c \in (0,1]$, as long as $\operatorname{Vol}\left(M - B\left(p,r\right)\right) \geq c^{3/2}\operatorname{Vol}\left(B\left(p,r\right)\right)$, we have

$$\frac{d}{dr}\left\{\operatorname{Vol}\left(B\left(p,r\right)\right)^{1/3}\right\} \geq \frac{1}{3}C_I c.$$

Since $\operatorname{Vol}\left(B\left(p,0\right)\right) = 0$, then

$$\operatorname{Vol}\left(B\left(p,r\right)\right) \geq \left(\frac{1}{3}C_I c\right)^3 r^3.$$

In particular, if $\operatorname{Vol}\left(B\left(p,r\right)\right) \leq \frac{1}{2}\operatorname{Vol}\left(M\right)$, so that we may take $c = 1$, then

$$\frac{\operatorname{Vol}\left(B\left(p,r\right)\right)}{r^3} \geq \frac{C_I^3}{27}.$$

LEMMA 6.31. *There exists a constant $\kappa > 0$ depending only on C_I such that if $r \in \left[0, \operatorname{Vol}\left(M\right)^{1/3}\right]$, then*

$$\frac{\operatorname{Vol}\left(B\left(p,r\right)\right)}{r^3} \geq \kappa.$$

PROOF. If $r \in \left[0, \operatorname{Vol}\left(M\right)^{1/3}\right]$, then if also $\operatorname{Vol}\left(B\left(p,r\right)\right) \leq \frac{1}{2}r^3$, we have $\operatorname{Vol}\left(B\left(p,r\right)\right) \leq \frac{1}{2}\operatorname{Vol}\left(M\right)$, which implies

$$\frac{\operatorname{Vol}\left(B\left(p,r\right)\right)}{r^3} \geq \frac{C_I^3}{27}.$$

Hence for all $r \in \left[0, \operatorname{Vol}\left(M\right)^{1/3}\right]$, we have

$$\operatorname{Vol}\left(B\left(p,r\right)\right) \geq \min\left\{\frac{1}{2}, \frac{1}{27}C_I^3\right\} r^3.$$

□

6. Cross curvature flow

In this section, the ideas do not come from the study of surfaces, but we have situated the section here for lack of a better place. It is an interesting question to see if any of these ideas, which are based in dimension 3, generalize or are related to ideas that work in higher dimensions.

The Ricci tensor has the following nice property which follow directly from the contracted second Bianchi identity (see also [**256**], Corollary 3.2).

LEMMA 6.32 (Ricci tensor and harmonic identity map). *If* Rc *is positive (negative) definite, then the identity map*

$$\mathrm{id} : (M^n, g) \to (M^n, \pm \mathrm{Rc})$$

is a harmonic map.

Now consider dimension 3. At any point, there exists a orthonormal frame $\{e_i\}_{i=1}^3$ such that

$$\mathrm{Rm}\,(e_i, e_j)\, e_k = \sigma_{ij}\,(\delta_{jk} e_i - \delta_{ik} e_j) \tag{6.24}$$

where $\sigma_{ij} = \sigma_{ji}$ is the sectional curvature of the plane spanned by e_i and e_j. In an (oriented) orthonormal frame, the **cross curvature tensor** Cr is defined by

$$C_{ij} = \mathrm{Cr}\,(e_i, e_j) \doteqdot \frac{1}{2} d\mu_{ipq} d\mu_{jrs} \left(R_{pr} - \frac{1}{2} R g_{pr}\right)\left(R_{qs} - \frac{1}{2} R g_{qs}\right),$$

where $d\mu$ is the volume form ($d\mu_{ijk}$ is nonzero only when i, j, k are distinct, in which case it is the sign of the permutation (ijk).) In an orthonormal frame satisfying (6.24), where $\kappa_1 \doteqdot R_{2332}$, $\kappa_2 \doteqdot R_{1331}$, and $\kappa_3 \doteqdot R_{1221}$, we have Rc and Cr are diagonal and $\mathrm{Rc}\,(e_i, e_i) = \kappa_j + \kappa_k$ and $\mathrm{Cr}\,(e_i, e_i) = \kappa_j \cdot \kappa_k$, where i, j, k are distinct.

LEMMA 6.33 (Cross curvature tensor and dual harmonic identity map). *If the sectional curvatures of* (M^3, g) *are either positive or negative (so that* $\mathrm{Cr} > 0$*), then*

$$\left(\mathrm{Cr}^{-1}\right)^{ij} \nabla_i C_{jk} = \frac{1}{2}\left(\mathrm{Cr}^{-1}\right)^{ij} \nabla_k C_{ij}, \tag{6.25}$$

and hence the identity map

$$\mathrm{id} : (M^n, \mathrm{Cr}) \to (M^n, g)$$

is a harmonic map.

Comparing Lemmas 6.32 and 6.33, we may think of Rc and Cr as being dual to each other. The **cross curvature flow** is defined on a 3-manifold of negative (positive) sectional curvature by

$$\frac{\partial}{\partial t} g_{ij} = \pm 2 C_{ij}.$$

Using DeTurck's trick, J. Buckland has shown that if $\left(M^3, g\right)$ is closed, then for any smooth initial metric with definite sectional curvature, a solution to the cross curvature flow exists for a short time [**59**].

CONJECTURE 6.34 (Chow-Hamilton - convergence of the cross curvature flow). *If $\left(M^3, g\right)$ is a closed 3-manifold with negative (positive) sectional curvature, then a solution $g(t)$ to the normalized cross curvature flow*

$$\frac{\partial}{\partial t} g_{ij} = \pm 2C_{ij} \mp \frac{2c}{3} g_{ij},$$

where $c \doteqdot \int_{M^3} g^{ij} C_{ij} d\mu / \int_{M^3} d\mu$, with $g(0) = g_0$ exists for all $t \geq 0$ and converges to a constant negative (positive) sectional curvature metric as $t \to \infty$.

If the initial metric has negative sectional curvature, then we have the following monotonicity formulas. Let $E_{ij} \doteqdot R_{ij} - \frac{1}{2} R g_{ij}$ be the Einstein tensor and define $T_{ijk} \doteqdot E_{i\ell} \nabla_\ell E_{jk}$ and $T_i \doteqdot \left(E^{-1}\right)^{jk} T_{ijk} = E_{ij} \nabla_j \log \frac{\det E}{\det g}$. Analogous to the decomposition of $\nabla \operatorname{Rc}$ (equation (3.36)), we decompose T_{ijk} into its irreducible components

$$T_{ijk} \doteqdot U_{ijk} - \frac{1}{10}\left(E_{ij} T_k + E_{ik} T_j\right) + \frac{2}{5} E_{jk} T_i,$$

where the coefficients $-\frac{1}{10}$ and $\frac{2}{5}$ are chosen make U traceless with respect to E : $\left(E^{-1}\right)^{ij} U_{ijk} = \left(E^{-1}\right)^{ik} U_{ijk} = \left(E^{-1}\right)^{jk} U_{ijk}$, we obtain

$$\left|T_{ijk} - T_{jik}\right|^2 = \left|U_{ijk} - U_{jik}\right|^2 + \left|T_i\right|^2.$$

Here all of the norms are with respect to E, so that for instance $\left|T_i\right|^2 \doteqdot \left(E^{-1}\right)^{ij} T_i T_j$.

PROPOSITION 6.35 (Two monotonicity formulas). *As long as a solution to the cross curvature flow exists (see [**151**]):*

$$\frac{d}{dt} \operatorname{Vol}(E) = \frac{1}{4} \int_{M^3} \left|U_{ijk} - U_{jik}\right|^2 \left(\frac{\det E}{\det g}\right)^{1/2} d\mu \geq 0 \tag{6.26}$$

and

$$\begin{aligned} \frac{d}{dt} \int_{M^3} & \left(\frac{1}{3} \operatorname{Trace}_g E - \left(\frac{\det E}{\det g}\right)^{1/3}\right) d\mu \\ &= -\frac{1}{6} \int_{M^3} \left(\left|U_{ijk} - U_{jik}\right|^2 + \frac{1}{3}\left|T_i\right|^2\right) \left(\frac{\det E}{\det g}\right)^{1/3} d\mu \\ &\quad - \int_{M^3} \left(\frac{\operatorname{Trace}_g \operatorname{Cr}}{3} - \left(\frac{\det \operatorname{Cr}}{\det g}\right)^{1/3}\right) \left(\frac{\det E}{\det g}\right)^{1/3} d\mu \\ &\leq 0. \end{aligned} \tag{6.27}$$

Note that $\mathrm{Vol}(E)$ is scale-invariant in g and that the integrand in (6.27) is nonnegative since it is the difference between the **arithmetic** and **geometric means** of $-\kappa_1$, $-\kappa_2$, and $-\kappa_3$. This difference vanishes if and only if the sectional curvature is constant. Note that the metric g is expanding and the integral scales like $g^{1/2}$, so the monotonicity formula (6.27) is somewhat surprising.

REMARK 6.36. *Ben Andrews* [**21**] *has observed that if the universal cover of* (M^3, g_0) *can be isometrically embedded as a hypersurface in Euclidean or Minkowski space, then the Gauss curvature flow of the hypersurface yields the cross curvature flow of the induced metric. When, in addition,* M^3 *is closed, Andrews has stated that global existence and convergence holds.*

7. Notes and commentary

§4. There is also a Harnack estimate for the scalar curvature under the additional condition of positive Ricci curvature [**134**]. The Yamabe flow for initial metrics with large energies has been studied by Schwetlick and Struwe [**449**].

§6. Every closed manifold of dimension at least 3 admits a metric with negative Ricci curvature. In dimension 3 this was proved by Gao and Yau [**220**]. In all dimensions ≥ 3 this was proved by Lohkamp [**351**] (see also [**352**]). Lohkamp showed that any smooth metric on a closed manifold may be approximated in C^0 by metrics with negative Ricci curvature.

Considering a much stronger curvature condition, Gromov and Thurston [**244**] showed that for every $\varepsilon > 0$, there exists a closed manifold (M^n, g) of dimension $n \geq 4$ with

$$-1 - \varepsilon \leq \mathrm{sect}(g) \leq 1$$

and which does not admit a hyperbolic metric.

PROBLEM 6.37. *Given* $n \geq 4$, *does there exist* $\varepsilon > 0$ *such that if a closed Riemannian manifold* (M^n, g) *has* $-1 - \varepsilon \leq \mathrm{sect}(g) \leq 1$, *then* M^n *admits an Einstein metric with negative sectional curvature.*

For work related to this direction using the Ricci flow see Ye [**532**]. However, by the work of Farrell and Ontaneda [**205**], it is not in general possible to use the Ricci flow (or other flow with continuous dependence on the initial metric) to evolve negative sectional curvature pinched metrics to Einstein metrics with negative sectional curvature. This is because in general, there is no continuous map from the space of negative sectional curvature pinched metrics to the space of Einstein metrics with negative sectional curvature.

CHAPTER 7

Introduction to singularities

One of the main topics of interest in Ricci flow and other geometric evolution equations is that of singularity formation. Here one would like to understand the possible singularities which arise in finite and infinite time singularities as well as the structure of solutions in high curvature regions. In this chapter we introduce the reader to this topic, focusing more on classical techniques. In Volume 2 we shall go into a more detailed study of singularities based on Perelman's work building upon Hamilton's earlier work on singularity formation. We begin this chapter by looking at how to dilate about a singularity. We then recall the classification of types of singularities based on the rate of blow up of the curvature. An interesting, but presumably non-generic singularity is the so-called degenerate neck pinch. Here one has a heuristic argument for why such a singularity should exist. Since singularity models are ancient solutions, we consider the classification of ancient solutions on surfaces. Furthermore, we survey some of the classical results on 3-dimensional singularity models; in Volume 2 we shall consider this topic in more detail. Finally, we present some questions and conjectures about ancient solutions in low dimensions.

1. Dilating about a singularity and taking limits

Now let's assume $(M^n, g(t))$ is a solution on a closed manifold on a maximal time interval $[0,T)$, where $T < \infty$, so that $\sup_{M\times[0,T)} |\mathrm{Rm}| = \infty$. We try to understand the singularity which is forming as $t \to T$ by considering sequences of points and times $\{(x_i, t_i)\}_{i\in\mathbb{N}}$ such that $x_i \in M$ and $t_i \to T$. We call the points x_i the origins. From the single solution $(M^n, g(t))$ we form a sequence of solutions $(M^n, g_i(t))$ defined by

$$\boxed{g_i(t) = K_i g\left(t_i + K_i^{-1} t\right)} \tag{7.1}$$

where $K_i \doteqdot |\mathrm{Rm}(x_i, t_i)|$. We are interested in the sequences where

$$\lim_{i\to\infty} K_i = \infty.$$

There are various conditions which we can put on the sequence $\{(x_i, t_i)\}$ to ensure that we obtain either a global or a local limit. We start with the classical point of view where we first assume there exists a positive constant

$C < \infty$ such that

$$K_i \geq \frac{1}{C} \sup_{M^n \times \left[t_i - \beta_i K_i^{-1}, t_i\right]} |\mathrm{Rm}| \tag{7.2}$$

where $\beta_i \to \infty$. In this case, we have

$$\sup_{M^n \times [-\beta_i, 0]} |\mathrm{Rm}\,(g_i)| \leq C.$$

That is, we have a uniform curvature bound for the sequence of solutions on time intervals ending at 0 and beginning at times which tend to $-\infty$. In order to apply Hamilton's Cheeger-Gromov compactness theorem (see Theorem 5.19) to a get a limit solution on an n-dimensional manifold, we need an injectivity radius estimate at the origins of the form:

$$\mathrm{inj}_{\,g_i(0)}\,(x_i) \geq \iota > 0$$

for some $\iota > 0$. This is equivalent to $\mathrm{inj}_{\,g(t_i)}\,(x_i) \geq K_i^{-1/2} \iota$.

Remark 7.1. *Given a constant* $1 < C < \infty$, *let*

$$M_C \doteqdot \left\{ (x,t) : |\mathrm{Rm}\,(x,t)| \geq \frac{1}{C} \sup_{M^n} |\mathrm{Rm}\,(g\,(t))| \right\}.$$

Given a sequence $\{(x_i, t_i)\}$ *with* $t_i \to T$, *we think of* $\{(x_i, t_i)\}$ *as being at the* ***smallest space scale*** *(**largest curvature scale**) if there exists* $C < \infty$ *such that* $(x_i, t_i) \in M_C$.

The following local injectivity radius estimate is a corollary of Perelman's no local collapsing theorem which we shall discuss in Volume 2.

Corollary 7.2 (Injectivity radius estimate at smallest space scale). *Let* $c > 0$ *and* $C < \infty$ *be constants. There exists a constant* $\iota > 0$ *depending only on* c, C *and* g_0 *such that if* $\{(x_i, t_i)\}$ *is any sequence of points and times with* $t_i < T < \infty$ *such that* $|\mathrm{Rm}\,(g\,(t_i))| \leq C K_i$ *in the ball* $B_{g(t_i)}\left(x_i, c K_i^{-1/2}\right)$, *then* $\mathrm{inj}_{\,g(t_i)}\,(x_i) \geq K_i^{-1/2} \iota$.

With this, we can apply the compactness theorem to get:

Theorem 7.3 (Ancient limit). *If* $\{(x_i, t_i)\}$ *is a sequence satisfying (7.2) for some* $C < \infty$, *then there exists a subsequence such that the sequence of pointed solutions to Ricci flow* $\{(M^n, g_i\,(t)\,, x_i)\}$ *converges uniformly in every* C^k *norm on compact sets to a complete solution* $(M^n, g_\infty\,(t)\,, x_\infty)$ *to the Ricci flow on the time interval* $(-\infty, 0]$.

Such a solution $(M^n, g_\infty\,(t))\,, t \in (-\infty, 0]$, is called a **singularity model**. Ancient solutions with bounded curvature have nonnegative scalar curvature:

Lemma 7.4 (Ancient solutions have nonnegative scalar curvature). *If* $(M^n, g\,(t))\,,\ t \in (-\infty, 0]$, *is a complete solution to the Ricci flow with curvatures bounded on each compact time interval, then either* $R\,(g\,(t)) > 0$ *for all* $t \in (-\infty, 0]$ *or* $\mathrm{Rc}\,(g\,(t)) \equiv 0$ *for all* $t \in (-\infty, 0]$.

PROOF. *(Idea.)* From (2.2) and $|\mathrm{Rc}|^2 \geq \frac{1}{n}R^2$, we have $\frac{\partial}{\partial t}R \geq \Delta R + \frac{2}{n}R^2$. As long as we can apply the maximum principle (see [**143**], Chapter 4 for why we can), since the solution exists on any interval $[\alpha, 0]$, we have $R(x,t) \geq -\frac{n}{2(t-\alpha)}$ for all $t \in (\alpha, 0]$. Taking the limit as $\alpha \to -\infty$, we conclude $R(x,t) \geq 0$ for all $t \in (-\infty, 0]$. The strong maximum principle implies either $R > 0$ always or $R \equiv 0$ always. In the latter case, by the evolution equation for R, we deduce $\mathrm{Rc} \equiv 0$. □

2. Singularity types

It is useful to divide *finite time* singularities into two types. Suppose that $(M^n, g(t))$ is a solution to the Ricci flow on a closed manifold on a maximal time interval $[0, T)$. We say that the solution forms a:

(1) **Type I singularity** (rapidly forming) if $\sup_{M\times[0,T)} (T-t)|\mathrm{Rm}| < \infty$,
(2) **Type IIa singularity** (slowly forming) if $\sup_{M\times[0,T)} (T-t)|\mathrm{Rm}| = \infty$.

Similarly, we may divide *infinite time* singularities:

(1) **Type III singularity** (rapidly forming) if $\sup_{M\times[0,\infty)} t|\mathrm{Rm}| < \infty$,
(2) **Type IIb singularity** (slowly forming) if $\sup_{M\times[0,\infty)} t|\mathrm{Rm}| = \infty$.

PROBLEM 7.5. *Under what conditions can one determine the blow up rate of Type IIa solutions? Perhaps one can begin by considering rotationally symmetric solutions.*

One difficulty could be that Type IIa singularities are not expected to be generic. In contrast, most immersed convex plane curves with winding number 2 develop Type II singularities under the curve shortening flow. Moreover, Angenent and Velazquez [**26**] proved that for solutions the curve shortening flow of convex immersed plane curves symmetric about the x-axis and with exactly one singular arc of turning angle π, the maximum curvature satisfies the following asymptotics:

$$\boxed{\max_{\gamma(t)} k = (1 + o(1))\sqrt{\frac{\log(\log[1/(T-t)])}{T-t}}.}$$

In comparison, a Type I singularity of the curve shortening flow is defined by:

$$\max_{\gamma(t)} k \leq \frac{C}{\sqrt{T-t}}$$

where C is independent of t.

2.1. Curvature gaps. We have the following lower bound for the blow up rate of the curvature of a finite time singularity.

LEMMA 7.6 (Curvature gap estimate for finite time singular solutions). *If* $(M^n, g(t))$ *is a solution to the Ricci flow on a closed manifold on a maximal time interval* $[0,T)$, *where* $T < \infty$, *then*

$$\boxed{(T-t)\max_{x\in M^n}|\mathrm{Rm}(x,t)| \geq \frac{1}{8}.}$$

PROOF. Recall from (5.1) that

$$\frac{\partial}{\partial t}|\mathrm{Rm}|^2 \leq \Delta|\mathrm{Rm}|^2 + 16|\mathrm{Rm}|^3 .$$

By the maximum principle, the quantity $K(t) \doteqdot \max_{x\in M^n}|\mathrm{Rm}(x,t)|^2$ satisfies $dK/dt \leq 16K^{3/2}$. From this and $\lim_{t\to T} K(t)^{-1/2} = 0$, we conclude that

$$K(t)^{-1/2} \leq 8(T-t)$$

and the lemma follows. □

For infinite time singularities, provided we have an injectivity radius estimate at each time for some point, we have a similar bound for the maximum curvature.

LEMMA 7.7 (Curvature gap estimate for immortal solutions). *If* $(M^n, g(t))$ *is a solution to the Ricci flow on a closed manifold on the infinite time interval* $[0,\infty)$,

$$\max_{x\in M^n}|\mathrm{Rm}(x,t)| \leq Ct^{-\delta}$$

for some $\delta > 0$, *and if*

$$\max_{x\in M^n}\mathrm{inj}(x,t)^2 \cdot \max_{x\in M^n}|\mathrm{Rm}(x,t)| \geq \iota$$

for some $\iota > 0$, *then there exists a constant* $c > 0$ *(depending on* δ *and* n*) such that*

$$\limsup_{t\to\infty}\left(t\max_{x\in M^n}|\mathrm{Rm}(x,t)|\right) \geq c. \tag{7.3}$$

PROOF. The length of a fixed path $\gamma : [a,b] \to M^n$ (with positive speed) evolves by (6.22)

$$\frac{d}{dt}L_{g(t)}(\gamma) = -\int_\gamma \mathrm{Rc}(T,T)\,ds,$$

where $T = \frac{d\gamma/du}{|d\gamma/du|}$ is the unit tangent vector and $ds = |d\gamma/du|\,du$ is the arc length element. Suppose, for the sake of obtaining a contradiction, that

$$t\max_{x\in M^n}|\mathrm{Rc}(x,t)| \leq \varepsilon$$

for some $\varepsilon > 0$ to be chosen later and for t sufficiently large. Then

$$\frac{d}{dt}L_{g(t)}(\gamma) \leq \frac{\varepsilon}{t}L_{g(t)}(\gamma)$$

for t sufficiently large. This implies that the diameter satisfies

$$\frac{d}{dt}\operatorname{diam}(g(t)) \leq \frac{\varepsilon}{t}\operatorname{diam}(g(t)).$$

From this we conclude

$$\operatorname{diam}(g(t)) \leq Ct^{\varepsilon}$$

for some $C < \infty$. Then

$$\operatorname{diam}(g(t))^2 \max_{x \in M^n} |\mathrm{Rm}(x,t)| \leq Ct^{2\varepsilon - \delta} \to 0$$

as $t \to \infty$, provided $\varepsilon < \delta/2$. That is, the metrics approach **almost flat** in the sense of Gromov [**241**], [**66**]. Since for any metric g, $\max_{x \in M^n} \operatorname{inj}_g(x) \leq \operatorname{diam}(g)$, we conclude that

$$\max_{x \in M^n} \operatorname{inj}(x,t)^2 \cdot \max_{x \in M^n} |\mathrm{Rm}(x,t)| \to 0,$$

which is a contradiction to our assumption. □

PROBLEM 7.8. *Can we remove the injectivity radius assumption in the above lemma? Note that the homogeneous solutions on* Nil *in Chapter 4, §10 satisfy (7.3).*

2.2. Elementary point picking.

2.2.1. *Type I.* If we have a Type I singular solution, then it is natural to choose any sequence of points and times (x_i, t_i) with $t_i \to T$ and

$$(T - t_i)\,|\mathrm{Rm}(x_i, t_i)| \geq c \tag{7.4}$$

for some $c > 0$. Recall that the dilated solutions $g_i(t)$ are defined by (7.1):

$$g_i(t) = K_i g\left(t_i + K_i^{-1} t\right)$$

on the time interval $[-t_i K_i, (T - t_i) K_i)$, where $K_i = |\mathrm{Rm}(x_i, t_i)|$. They satisfy the curvature estimates

$$\begin{aligned}
|\mathrm{Rm}(g_i(t))| &= K_i^{-1}\left|\mathrm{Rm}\left(g\left(t_i + K_i^{-1} t\right)\right)\right| \\
&\leq K_i^{-1} \frac{C}{T - \left(t_i + K_i^{-1} t\right)} = \frac{C}{K_i(T - t_i) - t} \\
&\leq \frac{C}{c - t}
\end{aligned}$$

for $t \in [-t_i K_i, c)$, which are independent of i. By the compactness theorem, since we have an injectivity radius estimate from Perelman's no local collapsing theorem, the sequence of pointed solutions $(M^n, g_i(t), x_i)$ preconverges to a complete ancient solution $(M^n_\infty, g_\infty(t), x_\infty)$ defined on $(-\infty, c)$ with $|\mathrm{Rm}(g_\infty(t))| \leq \frac{C}{c-t}$; that is, it is Type I.

The prime examples of Type I singular solutions are:

(1) A closed 3-manifold with positive Ricci curvature or a closed 4-manifold with positive curvature operator. In this case, as the time approaches T, the manifold shrinks to a point while approaching constant sectional curvature. Since exponential convergence in C^∞ is known, it follows that for any sequence of points and times (x_i, t_i) with $t_i \to T$ we must have $(T - t_i) |\mathrm{Rm}(x_i, t_i)| \geq c$ for some $c > 0$. The limit ancient solution $(M_\infty^n, g_\infty(t), x_\infty)$ is a shrinking spherical space form with $|\mathrm{Rm}(g_\infty(0))| \equiv 1$ and is defined on $(-\infty, \omega)$, where $\omega > 0$ is the singular time.

(2) A neck pinch forming on a 3-manifold. An idealized example of this is the round cylinder $S^2 \times S^1$ where the radius of S^2 at time t is $r(t) = \sqrt{r_0^2 - t}$ and the radius of S^1 is independent of t. In this case, for any sequence of points and times (x_i, t_i) with $t_i \to T$, the corresponding sequence of solutions $(M^n, g_i(t), x_i)$ converges to the ancient solution $\left(S^2 \times \mathbb{R}, g_\infty(t)\right)$, where S^2 has radius $r(t) = \sqrt{2 - t}$ since $R(g_\infty(0)) = |\mathrm{Rm}(g_\infty(0))| \equiv 1$.

REMARK 7.9 (Parabolic rescaling). *Given a singular solution* $(M^n, g(t))$, $t \in [0, T)$, *where* $T < \infty$, *consider the rescaled family of metrics*

$$\tilde{g}(\tau) \doteqdot e^\tau g\left(T - e^{-\tau}\right),$$

where $\tau \in (-\log T, \infty)$. *We easily compute that*

$$\frac{\partial}{\partial \tau} \tilde{g}(\tau) = -2 \operatorname{Rc}(\tilde{g}(\tau)) + \tilde{g}(\tau).$$

Given a sequence of points $x_i \in M^n$ *and times* $\tau_i \to \infty$, *we may consider the sequence of pointed time-dependent manifolds* $(M^n, \tilde{g}(\tau + \tau_i), x_i)$. *This is essentially the same as considering the dilated sequence* $g_i(t)$ *given by (7.1) and where* $t_i \doteqdot T - e^{-\tau_i}$.

2.2.2. *Type IIa.* For Type IIa singular solutions we shall assume that M^n is closed.[1] Assume that

$$\sup_{M \times [0,T)} (T - t) R = \infty. \tag{7.5}$$

The reason we use R instead of Rm is related to applying the Harnack inequality; in particular, see Theorem 8.36. In dimension 3, for solutions to the Ricci flow on closed manifolds, $|\mathrm{Rm}| \leq CR + C$ for some $C < \infty$, so that (7.5) is equivalent to the Type IIa condition. However, in higher dimensions this is not true in general and one should replace R by $|\mathrm{Rm}|$ in the following arguments.

Unlike the Type I case, we choose the sequence of points and times more carefully. Roughly speaking, we want to choose the points and times so that they maximize $(T - t) R(x, t)$. However, by assumption the supremum of this is ∞. Hence we maximize over time intervals approaching the maximal

[1] For the noncompact case we refer the reader to §16 of [**267**] or §8.4.2 of [**153**].

time interval. So first choose any sequence of times $T_i \nearrow T$ and then choose points and times $(x_i, t_i) \in M^n \times [0, T_i]$ such that

$$(T_i - t_i) R(x_i, t_i) = \max_{M^n \times [0,T_i]} (T_i - t) R(x, t). \tag{7.6}$$

Let $R_i \doteqdot R(x_i, t_i)$ and

$$g_i(t) \doteqdot R_i g\left(t_i + R_i^{-1} t\right). \tag{7.7}$$

Then by (7.6)

$$\begin{aligned} R(g_i)(x, t) &= R_i^{-1} R(g)\left(x, t_i + R_i^{-1} t\right) \\ &\leq \frac{(T_i - t_i) R_i}{(T_i - t_i) R_i - t} \end{aligned} \tag{7.8}$$

for all $x \in M^n$ and $t \in [-t_i R_i, (T_i - t_i) R_i)$, where $(T_i - t_i) R_i \to \infty$. Assuming the estimate $|\mathrm{Rm}| \leq CR + C$, we may apply Perelman's no local collapsing theorem, the higher derivative estimates and the compactness theorem, to see that the sequence $(M^n, g_i(t), x_i)$ subconverges to a complete limit $(M_\infty^n, g_\infty(t), x_\infty)$, which is defined for all $t \in (-\infty, \infty)$. By (7.8) and the fact that for any $t \in (-\infty, \infty)$, $\lim_{N \to \infty} \frac{N}{N-t} = 1$, the limit satisfies the curvature estimate

$$\max_{M_\infty^n \times (-\infty,\infty)} R(g_\infty)(x, t) \leq 1 = R(g_\infty)(x_\infty, 0). \tag{7.9}$$

The main conjectured example of a Type IIa singular solution is the degenerate neck pinch described in the next section.

EXERCISE 7.10. *When M^n is noncompact and the solution has bounded curvature at each time, we may choose $(x_i, t_i) \in M^n \times [0, T_i]$ so that*

$$(T_i - t_i) R(x_i, t_i) \geq (1 - \varepsilon_i) \sup_{M^n \times [0,T_i]} (T_i - t) R(x, t)$$

where $\varepsilon_i \to 0$. Show that if the limit $g_\infty(t)$ of $g_i(t)$ exists (the no local collapsing theorem does not apply since M^n is noncompact), then $g_\infty(t)$ satisfies (7.9).

2.2.3. *Type IIb.* Take any sequence $T_i \to \infty$ and choose $(x_i, t_i) \in M^n \times (0, T_i)$ such that

$$t_i (T_i - t_i) R(x_i, t_i) = \sup_{M^n \times [0,T_i]} (t (T_i - t) R(x, t)).$$

Note that the graph of the function $t \mapsto t(T_i - t)$ is an upside down parabola vanishing at 0 and T_i. As in the Type IIa case, define $g_i(t) \doteqdot R_i g\left(t_i + R_i^{-1} t\right)$ where $R_i \doteqdot R(x_i, t_i)$. Let $\alpha_i \doteqdot -t_i R_i$ and $\omega_i \doteqdot (T_i - t_i) R_i$ so that under dilation, $[\alpha_i, \omega_i]$ corresponds to $[0, T_i]$. We have

$$\begin{aligned} R(g_i)(x,t) &= R_i^{-1} R(g)\left(x, t_i + R_i^{-1} t\right) \\ &= \frac{\left(t_i + R_i^{-1} t\right)\left(T_i - t_i - R_i^{-1} t\right) R\left(x, t_i + R_i^{-1} t\right)}{t_i (T_i - t_i) R(x_i, t_i)} \\ &\times \frac{t_i R_i (T_i - t_i) R_i}{(t_i R_i + t)((T_i - t_i) R_i - t)} \\ &\leq \frac{\alpha_i}{\alpha_i - t} \frac{\omega_i}{\omega_i - t} \end{aligned}$$

for all $x \in M^n$ and $t \in [\alpha_i, \omega_i]$. Now

$$\begin{aligned} \frac{1}{-\alpha_i^{-1} + \omega_i^{-1}} &= \frac{\alpha_i \omega_i}{\alpha_i - \omega_i} = \frac{t_i (T_i - t_i) R_i}{T_i} \\ &= T_i^{-1} \sup_{M^n \times [0, T_i]} (t (T_i - t) R(x,t)) \\ &\geq \frac{1}{2} \sup_{M^n \times [0, T_i/2]} t R(x,t) \to \infty \end{aligned}$$

as $i \to \infty$. Hence $\alpha_i \to -\infty$ and $\omega_i \to \infty$. Hence there exists a subsequence such that $(M^n, g_i(t), x_i)$ converges in C^∞ on compact sets to a complete solution $(M^n_\infty, , g_\infty(t), x_\infty)$, $t \in (-\infty, \infty)$, with

$$\sup_{M^n_\infty \times (-\infty, \infty)} R(g_\infty) = 1 = R(g_\infty)(x_\infty, 0).$$

PROBLEM 7.11. *Given a solution to the Ricci flow on a closed manifold, is it possible that a Type IIb singularity exists? We are not aware of any existence result for Type IIb singularities of the Ricci flow on closed manifolds.*

2.2.4. *Type III.* For Type III singular solutions, $\sup_{M^n \times [0,\infty)} t |\mathrm{Rm}(g(t))| \doteqdot A < \infty$. Analogous to the Type I case, choose any sequence of points and times (x_i, t_i) with $t_i \to \infty$ and

$$t_i |\mathrm{Rm}(x_i, t_i)| \geq c \tag{7.10}$$

for some $c > 0$. Again $g_i(t) \doteqdot K_i g\left(t_i + K_i^{-1} t\right)$, $t \in [-t_i K_i, \infty)$, where $K_i = |\mathrm{Rm}(x_i, t_i)|$. We have the Type III curvature bound:

$$\begin{aligned} |\mathrm{Rm}(g_i(t))| &= K_i^{-1} \left|\mathrm{Rm}\left(g\left(t_i + K_i^{-1} t\right)\right)\right| \\ &\leq \frac{A}{t_i K_i + t} \leq \frac{A}{c + t} \end{aligned}$$

for $t \in (-t_i K_i, \infty)$. Since Perelman's no local collapsing theorem only applies to finite time singularities, we do not know if a subsequence converges.

Examples of Type III singularities are expanding hyperbolic manifolds and most of the homogeneous solutions on 3-manifolds as partially discussed in §10 of Chapter 4.

2.3. Curvature bounds for ancient solutions. The following result is the 'ancient' analogue of Lemma 7.6.

LEMMA 7.12 (Curvature gap estimate for ancient solutions). *If $(M^n, g(t))$ is an ancient solution with bounded nonnegative Ricci curvature which is not Ricci flat , then*

$$\liminf_{t\to-\infty} |t| \sup_{x\in M^n} R(x,t) \geq \frac{1}{2}.$$

PROOF. Since $\mathrm{Rc} \geq 0$, we have

$$\frac{\partial}{\partial t} R = \Delta R + 2|\mathrm{Rc}|^2 \leq \Delta R + 2R^2.$$

Hence $R_{\max}(t) \doteqdot \sup_{x\in M^n} R(x,t) > 0$ satisfies

$$\frac{d}{dt}(R_{\max}(t))^{-1} \geq -2$$

and the lemma follows easily. □

REMARK 7.13. *We have actually shown that there exists a constant $C < \infty$ depending on the solution such that*

$$R_{\max}(t) \geq \frac{1}{R_{\max}(0)^{-1} - 2t}$$

for all $t \leq 0$. Basically this says that the maximum scalar curvature of a non-Ricci flat ancient solution with bounded nonnegative Ricci curvature cannot decay faster than comparable to that of an expanding sphere as we go backward in time toward $-\infty$.

Recall that if we have a complete ancient solution with bounded (with the bound possibly depending on time) nonnegative curvature operator, then by the trace Harnack inequality (8.49), we have $\frac{\partial R}{\partial t} \geq 0$. In particular, for all $(x,t) \in M^n \times (-\infty, 0]$, we have

$$R(x,t) \leq C \doteqdot \sup_{y\in M} R(y,0).$$

3. Degenerate neck pinch

To the best of our knowledge, it has not been shown that a Type IIa singularity can form under the Ricci flow on a closed manifold. We now describe a Type IIa singularity which is conjectured to exist, the degenerate neck pinch. To see how this singularity should arise, consider a 1-parameter family of rotationally symmetric solutions $g_s(t)$ to the Ricci flow with positive scalar curvature on a topological n-sphere, $n \geq 3$, parametrized by the unit interval: $s \in [0,1]$. When $s = 0$, let the initial metric $g_0(0)$ be a rotationally symmetric dumbbell, which is also invariant under a reflection, with two equally sized spherical regions joined by a thin neck such that the neck pinches off at the center after a short time. See Angenent and Knopf [**24**] (and [**469**] for the noncompact case) for sufficient conditions for such a neck to pinch. On the other hand, when $s = 1$, let the initial metric $g_1(0)$ be a round sphere, so that $g_1(t)$ shrinks homothetically to a round point. Now let the initial metrics $g_s(0)$ be a smooth 1-parameter family of rotationally symmetric metrics joining $g_0(0)$ to $g_1(0)$ with positive scalar curvature. We leave it to the reader to show that one can construct such families of solutions. Since all of the solutions are rotationally symmetric and topologically S^n, there are two 'tips' (at the ends) which are the stabilizers of the $\mathrm{SO}(n)$ action.

Since the solutions have positive scalar curvature, for each $s \in [0,1]$, the solution exists up to a finite time $T_s < \infty$ when a singularity forms. By the continuous dependence of the solution on the initial metric, we expect that there is some parameter $s_0 \in (0,1)$ such that for all $s \in [0, s_0)$, the solution $g_s(t)$ pinches off a neck at time T_s, whereas the solution $g_{s_0}(t)$ does not pinch off a neck at time T_{s_0} but rather forms another type of singularity. We *expect* that if we take a sequence of times $t_i \to T_{s_0}$ and dilate the solution $g_{s_0}(t)$ about (x_i, t_i) where x_i is a tip at which the curvature of $g_{s_0}(t_i)$ is equal to the maximum of the two tips, then we obtain the Bryant soliton as the corresponding limit solution.

We can also ask the following question.

PROBLEM 7.14. *Given a closed manifold on a topological spherical space form and a one-parameter family of initial metrics $g_s(0)$, $s \in [0,1]$ such that $g_0(t)$ forms a Type I singularity model isometric to a shrinking spherical space form S^n/Γ, and $g_1(t)$ forms a Type I singularity model isometric to a shrinking quotient of a cylinder $\left(S^{n-1} \times \mathbb{R}\right)/\Gamma$, then does there exist $s_0 \in (0,1)$ such that $g_{s_0}(t)$ forms a Type IIa singularity?*

The expectation is that at any $s_0 \in (0,1)$ where there is a transition in the kind of singularity model, the singularity type of $g_{s_0}(t)$ is IIa. Perhaps one can consider S^n where the one-parameter family of initial metrics are all rotationally symmetric. Here one expects that if a Type IIa singularity forms for one of the initial metrics, then the Bryant soliton is a singularity model of the solution.

3.1. Point picking for ancient solutions (backward limits). An ancient solution is **Type I** if

$$\boxed{\sup_{M^n \times (-\infty,0]} |t| \left|\mathrm{Rm}(x,t)\right| < \infty}$$

and **Type II** if

$$\boxed{\sup_{M^n \times (-\infty,0]} |t| \left|\mathrm{Rm}(x,t)\right| = \infty.}$$

As far as we know, there hasn't been an application of taking a backward limit in a Type I ancient solution (although there is a related idea when considering Hamilton's entropy to classify Type I ancient solutions on surfaces).

Similar to the point picking technique for Type IIa solutions (see §2.2.2), we can sometimes take a backward limit of a Type II ancient solution to obtain a steady gradient Ricci soliton. Here we take any sequence of times $T_i \to -\infty$ and choose $(x_i, t_i) \in M^n \times (T_i, 0)$ so that

$$\lim_{i\to\infty} \frac{|t_i|\,(t_i - T_i)\,|R(x_i, t_i)|}{\sup_{M^n \times [T_i, 0]} |t|\,(t - T_i)\,|R(x,t)|} = 1.$$

If one has an injectivity radius estimate, then one can show that the dilated solutions $(M^n, g_i(t), x_i)$ converge to a complete solution $(M^n_\infty, g_\infty(t), x_\infty)$ to the Ricci flow defined on the eternal time interval $(-\infty, \infty)$ with the property that $R(x_\infty, 0) = \sup_{M^3_\infty \times (-\infty,\infty)} R$. When we have, in addition, that $\operatorname{Rm}(g_\infty(t)) \geq 0$, we can conclude $(M^n_\infty, g_\infty(t))$ is a steady gradient Ricci soliton (see Theorem 8.36). A nice example of this is the choice of appropriate points and times in the Rosenau solution to obtain the cigar as a backward limit.

3.1.1. *The steady soliton solution as a limit of homothetic soliton (fundamental) solutions.* To get a better feel for backward limits, we consider the one-dimensional heat equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$$

on $\mathbb{R}$. A steady soliton solution is

$$g(x,t) \doteqdot e^{t-x}$$

which satisfies

$$g(x, t+s) = g(x-s, t).$$

We can obtain this steady soliton solution as the limit of homothetic soliton (fundamental) solutions of the form

$$f_{a,b,c}(x,t) = \frac{a}{\sqrt{t+c}} e^{-(x-b)^2/4(t+c)}.$$

In particular, consider the 1-parameter family of fundamental solutions

$$f_c(x,t) \doteqdot \frac{\sqrt{c}e^c}{\sqrt{t+c}} e^{-(x+2c)^2/4(t+c)}.$$

Another way to write this is:

$$f_c(x,t) = \left(\frac{c}{t+c}\right)^{1/2} \exp\left\{-\frac{1}{4}\frac{x^2 + 4c(x-t)}{t+c}\right\}.$$

From this we see

$$\lim_{c\to\infty} f_c(x,t) = e^{t-x}.$$

REMARK 7.15. *Note that we let $h(x,t) \doteqdot \frac{1}{\sqrt{t}} e^{-x^2/4t}$, then*

$$f_c(x,t) = f_{\sqrt{c}e^c, -2c, c}(x,t) = \sqrt{c}e^c h(x+2c, t+c).$$

4. Classification of ancient solutions on surfaces

A priori estimates for Ricci flow usually lead to geometric applications. The Harnack and entropy estimates are no exception. In this section we study ancient solutions on surfaces. Note that we study these ancient solutions in general and we do not assume they necessarily arise from dilating about a singularity of some solution to the Ricci flow. First we consider an application of the Harnack estimate.

THEOREM 7.16 (2-d eternal solutions attaining the sup of their curvatures are cigars). *If $\left(M^2, g\left(t\right)\right)$, $t \in (-\infty, \infty)$, is a complete solution to the Ricci flow with curvature bounded on compact time intervals and such that $\sup_{M\times(-\infty,\infty)} R$ is attained at some point in space and time, then $\left(M^2, g\left(t\right)\right)$ is either flat or isometric to a constant multiple of the cigar soliton solution.*

PROOF. By Lemma 7.4, we have either $g\left(t\right)$ is flat or has positive curvature. We claim that in the latter case $\left(M^2, g\left(t\right)\right)$ is a gradient Ricci soliton. The proposition then follows from this claim and Corollary 4.8.

To prove the claim, let

$$P \doteqdot \frac{\partial}{\partial t}\log R - \left|\nabla \log R\right|^2 = \Delta \log R + R.$$

This is the quantity defined in (8.20) without the $1/t$ term. One can compute that

$$\frac{\partial}{\partial t}P = \Delta P + 2\left\langle \nabla \log R, \nabla P\right\rangle + 2\left|\nabla\nabla \log R + \frac{1}{2}Rg\right|^2, \tag{7.11}$$

and hence

$$\frac{\partial}{\partial t}P \geq \Delta P + 2\left\langle \nabla \log R, \nabla P\right\rangle + P^2. \tag{7.12}$$

Here we applied the inequality $\left|a_{ij}\right|^2 \geq \frac{1}{n}\left(\operatorname{tr} a\right)^2$ to $a = \nabla\nabla \log R + \frac{1}{2}Rg$. Since the solution exists on the interval (α, ∞) for any $\alpha \in \mathbb{R}$, the maximum principle says

$$P = \Delta \log R + R \geq -\frac{1}{t-\alpha}$$

for all $t > \alpha$. Hence, at any $(x, t) \in M^2 \times (-\infty, \infty)$, by taking $\alpha \to -\infty$, we see that

$$\Delta \log R + R \geq 0.$$

Now by our hypothesis, there exists $(x_0, t_0) \in M^2 \times (-\infty, \infty)$ such that $R\left(x_0, t_0\right) = \sup_{M^2\times(-\infty,\infty)} R$. At (x_0, t_0) we have $\frac{\partial R}{\partial t} = 0$ and $|\nabla R| = 0$, and hence $P\left(x_0, t_0\right) = 0$. Since $P \geq 0$, applying the strong maximum principle to (7.11) we see that $P \equiv 0$ and hence $\nabla\nabla \log R + \frac{1}{2}Rg \equiv 0$ on $M^2 \times (-\infty, \infty)$, which says that $g(t)$ is a gradient Ricci soliton flowing along $\nabla \log R$. □

REMARK 7.17. *The above result extends to higher dimensions. See Theorem 8.36.*

Next we consider an application of the entropy estimate, Proposition **??**.

THEOREM 7.18 (2-d Type I ancient solutions are round 2-spheres). *If* $\left(M^2, g(t)\right)$, $t \in (-\infty, \omega)$, $\omega \leq \infty$, *is a solution to the Ricci flow on a complete surface with curvature bounded compact time intervals and such that* $\sup_{M\times(-\infty,\omega-1)} |t| R < \infty$, *then* $\omega < \infty$ *and the universal cover of* $\left(M^2, g(t)\right)$ *is isometric to either a round shrinking* S^2 *or the flat* $\mathbb{R}^2$.

REMARK 7.19. *For the noncompact case, see Proposition 7.20 (Theorem 26.1 of* [**267**].*)*

PROOF. As in Lemma 7.4, we have either $R \equiv 0$ or $R > 0$ everywhere; we assume the latter. By passing to the universal cover if necessary, we may assume $M^2 \cong S^2$.Since the area $\mathrm{A}(g(t))$ of M^2 evolves by

$$\frac{d}{dt}\mathrm{A}(g(t)) = -\int_{M^2} R\, d\mu = -8\pi,$$

we have $\mathrm{A}(g(t)) = 8\pi(T-t)$ for some constant T. The entropy is:

$$\begin{aligned} N(g(t)) &\doteqdot \int_{M^2} R \log(R\,\mathrm{A})\; d\mu \\ &= \int_{M^2} R \log[R(T-t)]\; d\mu + 8\pi \log(8\pi), \end{aligned}$$

which is a nonincreasing function of time. By our assumption, we have

$$\sup_{M\times(-\infty,\omega-1)} (T-t) R < \infty$$

and hence the limit

$$N_{-\infty} \doteqdot \lim_{t\to-\infty} N(g(t)) \tag{7.13}$$

exists and is finite.

Now take any sequence of points and times (x_i, t_i) with $R(x_i, t_i) = R_{\max}(t_i)$ and $t_i \to -\infty$, and consider the dilated solutions

$$g_i(t) \doteqdot K_i g\left(t_i + K_i^{-1} t\right), \tag{7.14}$$

where $K_i \doteqdot R(x_i, t_i)$. Since $0 < R(g_i(0)) \leq 1$, by Klingenberg's injectivity radius estimate (see Theorem 5.9 on p. 98 of [**103**]), $\mathrm{inj}(g_i(0)) \geq \sqrt{2}\pi$. Since $\mathrm{A}(g_i(0)) = 8\pi K_i (T - t_i) \leq C < \infty$, the diameters of $g_i(0)$ are uniformly bounded. Hence, by the Cheeger-Gromov type compactness Theorem 5.19, there is a subsequence $\left(M^2, g_i(t), x_i\right)$ which limits to an ancient solution

$$\left(M^2_{-\infty}, g_{-\infty}(t), x_{-\infty}\right)$$

of the Ricci flow on a *closed* surface of positive curvature. The entropy of the limit satisfies

$$N(g_{-\infty}(t)) = \lim_{i\to\infty} N(g_i(t)) = \lim_{i\to\infty} N\left(g\left(t_i + K_i^{-1}t\right)\right) \equiv N_{-\infty},$$

since $\lim_{i\to\infty}\left(t_i + K_i^{-1} t\right) = -\infty$ for all $t \in (-\infty, 0]$. In particular, the entropy of the limit is independent of time. By Corollaries **??** and **??**, we conclude that $g_{-\infty}(t)$ is a shrinking round 2-sphere. Since the constant curvature metrics minimize entropy among all metrics on S^2, we have $N_{-\infty} \leq N(g(t))$. But since $N(g(t))$ is nonincreasing, we also have $N_{-\infty} \geq N(g(t))$. Thus $N(g(t)) \equiv N_{-\infty}$ and $g(t)$ is a shrinking 2-sphere of constant curvature. □

PROPOSITION 7.20. *Any ancient solution* $\left(M^2, g(t)\right)$, $t \in (-\infty, \omega)$, *on a noncompact surface with* $R(g(t)) > 0$ *is Type II.*

REMARK 7.21. *That is, there does not exist a Type I ancient solution with positive curvature on a noncompact surface.*

PROOF 1. Suppose $\left(M^2, g(t)\right)$, $t \in (-\infty, \omega)$, is a Type I ancient solution on a noncompact surface with $R(g(t)) > 0$. If $\operatorname{ASCR}(g(t)) = \infty$, then we can apply the point picking methods to obtain an infinite number of bumps of curvature which contradicts Theorem 7.35.

Choose any origin $O \in M$. By Proposition 4.19 we have

$$\operatorname{AVR}(g) > 0$$

and there exists $c_1 > 0$ such that

$$R(x,t)\, d_{g(t)}(x,O)^2 \geq c_1$$

for all $x \in M^n$. Let

$$A(r) \doteqdot \left\{x \in M^2 : r \leq d(x,O) \leq 2r\right\}.$$

By the proof of Proposition 4.19 we can actually show that there exists $c_2 > 0$ such that

$$\operatorname{Area}(A(r)) \geq c_2 r^2.$$

Thus

$$\int_{A(r)} R d\mu \geq \frac{c_1}{r^2} \cdot c_2 r^2 = c_1 c_2 > 0.$$

Since $\left\{A\left(3^k\right)\right\}_{k=0}^{\infty}$ are disjoint, we conclude

$$\int_{M^2} R d\mu \geq \sum_{k=0}^{\infty} \int_{A\left(3^k\right)} R d\mu = \infty.$$

This contradicts the Cohn-Vossen inequality which says that

$$\int_{M^2} R d\mu \leq 4\pi.$$

□

Another proof is as follows.

PROOF 2. We claim $\operatorname{AVR}(g(t)) = 0$. For if $\operatorname{AVR}(g(t)) > 0$, then by [**400**] the ancient solution can be extended to a Type III immortal solution, that is, for $t \in (0, \infty)$ we have $tR(x,t) \leq C$ independent of x and t. By the Harnack inequality, since the solution is ancient, we also have $\frac{\partial R}{\partial t} \geq 0$ so that we conclude $R \equiv 0$. By Proposition 4.19, $\operatorname{ASCR}(g(t)) = \infty$, and as in the proof above, we obtain a contradiction to Theorem 7.35. $\square$

We conclude this section with what happens in the complementary case where $\sup_{M \times (-\infty, \omega - 1)} |t| R = \infty$:

Proposition 7.22 (2d - backward limit of Type II ancient solution is cigar). *If $(M^2, g(t))$, $t \in (-\infty, \omega)$, $\omega \leq \infty$, is a solution to the Ricci flow on a complete surface with curvature bounded on compact time intervals and such that*

$$\sup_{M \times (-\infty, \omega - 1)} |t| R = \infty,$$

then there exists a sequence of points and times (x_i, t_i) with $t_i \to -\infty$ such that $(M^2, g_i(t), x_i)$, where $g_i(t)$ is given by (7.14), limits to a constant multiple of the cigar soliton solution.

Proof. We first choose any times $T_i \to -\infty$ and positive numbers $\varepsilon_i \to 0$. Since the curvatures are uniformly bounded on compact time intervals, there exists a sequence (x_i, t_i) such that

$$|t_i| (t_i - T_i) R(x_i, t_i) \geq (1 - \varepsilon_i) \sup_{M \times [T_i, 0]} |t| (t - T_i) R(x, t). \tag{7.15}$$

Define

$$\alpha_i \doteqdot (t_i - T_i) R(x_i, t_i)$$
$$\omega_i \doteqdot -t_i R(x_i, t_i).$$

Then

$$\frac{1}{\alpha_i^{-1} + \omega_i^{-1}} = \frac{|t_i| (t_i - T_i) R(x_i, t_i)}{|T_i|} \to \infty$$

so that

$$\lim_{i \to \infty} \alpha_i = \infty = \lim_{i \to \infty} \omega_i.$$

Let $K_i \doteqdot R(x_i, t_i)$ and consider the sequence of dilated metrics

$$g_i(t) \doteqdot K_i g\left(t_i + K_i^{-1} t\right)$$

which are defined on the time interval $(-\infty, \omega_i]$. By (7.15) we have the curvature estimates

$$0 < R(g_i(t)) \leq \frac{\alpha_i \omega_i}{(1 - \varepsilon_i)(\alpha_i + t)(\omega_i - t)} \doteqdot f_i(t).$$

Since M^2 is noncompact, by the Gromoll-Meyer injectivity radius estimate, since $0 < K(g_i(0)) \leq \frac{1}{2(1-\varepsilon_i)}$,

$$\operatorname{inj}(g_i(0)) \geq \sqrt{2(1 - \varepsilon_i)} \pi.$$

Hence we can apply Hamilton's Cheeger-Gromov compactness theorem for solutions of Ricci flow to conclude that there exists a subsequence $(M^2, g_i(t), x_i)$ which converges to a limit solution $(M_\infty^2, g_\infty(t), x_\infty)$. Since on any compact time interval $I \subset (-\infty, \infty)$ we have $f_i(t) \to 1$ uniformly, this limit solution satisfies the curvature estimate

$$0 \leq R(g_\infty(t)) \leq 1$$

for all $t \in (-\infty, \infty)$. Since $R_{g_\infty(0)}(x_\infty) = \lim_{i\to\infty} R_{g_i(0)}(x_i) = 1 > 0$, the strong maximum principle implies $R(g_\infty(t)) > 0$. By Theorem 7.16, we conclude that $\left(M_\infty^2, g_\infty(t)\right)$ is a cigar soliton solution. □

In any dimension, an ancient solution $(M^n, g(t))$ is called an **ancient κ-solution** if it is complete, nonflat with bounded nonnegative curvature operator and if for any metric ball $B(x,r) \subset M^n$ with $r > 0$ and such that $|\operatorname{Rm}| \leq r^{-2}$ on $B(x,r)$, we have $\frac{\operatorname{Vol} B(x,r)}{r^n} \geq \kappa$ (see Chapter **??**, §**??** for further discussion.) Since the cigar soliton is not an ancient κ-solution for any $\kappa > 0$, from Propositions 7.18 and 7.22, we obtain the following.

Corollary 7.23 (Hamilton 1995 - $n = 2$, κ-ancient solutions are round S^2). *If $\left(M^2, g(t)\right)$ is an ancient κ-solution, then $\left(M^2, g(t)\right)$ is a round shrinking 2-sphere.*

Another proof of this will be given in Volume 2.

5. Extending noncompact ancient surface solutions to eternal solutions

In this section we prove a 2-dimensional analogue of Perelman's conjecture for ancient κ-solutions. This result hinges on an arbitrary dimensional long-time existence result for solutions of the Kähler-Ricci flow with nonnegative bisectional curvature and average scalar curvature decaying (see Volume 2 for the definition of bisectional curvature.)

Theorem 7.24 (Shi). *Let (M^n, g_0) be a complete noncompact Kähler manifold with bounded nonnegative bisectional curvature. Assume that there exists a $C > 0$ and $0 < \theta < 2$ such that for any $x \in M^n$ and $r > 0$,*

$$\frac{1}{\operatorname{Vol}(B(x,r))} \int_{B(x,r)} R \, d\mu \leq \frac{C}{(1+r)^\theta}. \tag{7.16}$$

Then a solution $g(t)$ to the Kähler-Ricci flow with $g(0) = g_0$ exists for all time.

See [**464**], Theorem 7.9 for the proof. There is a slightly more general result in [**404**] (Corollary 2.2) using a different and simpler approach of obtaining the key C^0 estimates. The result says that one has the same result if the integral on the left hand side of (7.16) is bounded by a positive function $k(r)$ with $\int_0^r sk(s) = o(r^2)$.

Theorem 7.25 (Ni - 2-d noncompact ancient solutions are eternal). *If $(M^2, g(t))$ is a 2-dimensional complete noncompact ancient solution to Ricci flow such that on any compact time interval the curvature is bounded, then $(M^2, g(t))$ can be extended to an eternal solution.*

PROOF. We assume that the solution exists on the time interval $(-\infty, 1]$ and consider a fixed time: $(M^2, g(0))$. Recall from Lemma 7.4 that if $(M^2, g(0))$ is not flat then $R > 0$. By the classical Cohn-Vossen inequality we know that

$$\int_{M^2} R(x,0)\, d\mu \leq 4\pi$$

The injectivity radius estimate of Gromoll-Meyer implies that the injectivity radius of $(M^2, g(0))$ has the lower bound $\pi/\sqrt{K}$, where $K \doteqdot \frac{1}{2}\sup_{x\in M^2} R(x,0)$. By Corollary 1.68, there exists $C = C(K)$ such that $\operatorname{Vol}(B(x,r)) \geq Cr$ for any $x \in M^2$ and $r \geq 1$. (Actually, since we have an injectivity radius estimate, in our case a simpler proof of the linear growth of the volume of balls follows from simply taking a ray and considering an infinite chain of disjoint unit balls centered at points along the ray.) We then have

$$\frac{1}{\operatorname{Vol}(B(x,r))}\int_{B(x,r)} R d\mu \leq \frac{C}{r+1}.$$

The result now follows from Theorem 7.24 (Riemannian surfaces are Kähler). □

This leads to the following question:

PROBLEM 7.26 (Are 2-d eternal solutions cigars?). *For every eternal solution on a surface with the curvature bounded at each time, is* $\sup_{M^2\times(-\infty,\infty)} R$ *finite and attained? If so, the solution must either be flat or the cigar soliton.*

6. Dimension reduction

Given a singular solution $(M^n, g(t))$ we are interested in singularity models $(M_\infty^n, g_\infty(t))$, which are the complete limit solutions of dilations of the original solution $g(t)$ about suitable sequences of points and times. In dimension 3, these singularity models have nonnegative sectional curvature and the classification result in section 4 of Chapter 4 applies. If the universal cover of $\left(M_\infty^3, g_\infty(t)\right)$ does not split, then $g_\infty(t)$ has positive sectional curvature. In this case one would like to study the **geometry at spatial infinity** of the solution $g_\infty(t)$ at a fixed time, say $t = 0$. To do this we shall now describe how to find a sequence of points $y_i \in M_\infty^3$ tending to spatial infinity about which to dilate the solution to get a limit which splits.

DEFINITION 7.27. *Let* (M^n, g) *be a complete noncompact Riemannian manifold with nonnegative Ricci curvature. An* ***asymptotic cone*** *of* (M^n, g) *is a pointed Gromov-Hausdorff limit of a sequence* $\{(M^n, \varepsilon_i g, p)\}_{i=1}^{\infty}$, *where* $\varepsilon_i \to 0$.

One can obtain interesting metrics on $\mathbb{R}^4$ invariant under $\mathrm{SU}(2)$ (for the following the reader may find it useful to first do Exercise 4.21).

EXERCISE 7.28 (Perelman [**415**]). *Let* $\{f_i\}_{i=1}^3$ *be left-invariant vector fields on* $\mathrm{SU}(2) \cong S^3$ *with*

$$[f_i, f_j] = 2f_k$$

where (ijk) *is a cyclic permutation of* (123). *Let* $\{\eta^i\}_{i=1}^3$ *denote the dual coframe field and consider a metric on* $\mathbb{R}^4$ *given in spherical coordinates by*

$$g = dr^2 + \sum_{i=1}^{3} a_i(r)^2 \, \eta^i \otimes \eta^i.$$

Show that if (ijk) *is a cyclic permutation of* (123), *then*

$$\left\langle R\left(f_i, \frac{\partial}{\partial r}\right)\frac{\partial}{\partial r}, f_i\right\rangle = -a_i(r)\, a_i''(r) = -\frac{a_i''(r)}{a_i(r)}\left|f_i\right|^2$$

$$\begin{aligned}&\langle R(f_i, f_j) f_j, f_i\rangle \\ &= |f_i|^2 |f_j|^2 \left(-\frac{a_i' a_j'}{a_i a_j} + \frac{a_i^4 + a_j^4 - 3a_k^4 + 2a_i^2 a_k^2 + 2a_j^2 a_k^2 - 2a_i^2 a_j^2}{a_i^2 a_j^2 a_k^2}\right)\end{aligned}$$

$$\begin{aligned}&\left\langle R(f_i, f_j) f_k, \frac{\partial}{\partial r}\right\rangle \\ &= \frac{|f_i|\,|f_j|\,|f_k|}{a_i a_j a_k}\left(-\frac{a_i'}{a_i}\left(a_k^2 + a_i^2 - a_j^2\right) + \frac{a_j'}{a_j}\left(a_i^2 - a_j^2 - a_k^2\right) + 2a_k a_k'\right)\end{aligned}$$

$$\begin{aligned}\left\langle R\left(f_i, \frac{\partial}{\partial r}\right)\frac{\partial}{\partial r}, f_j\right\rangle &= \langle R(f_i, f_j) f_j, f_k\rangle = \left\langle R\left(\frac{\partial}{\partial r}, f_i\right) f_i, f_j\right\rangle \\ &= \left\langle R\left(\frac{\partial}{\partial r}, f_i\right) f_i, f_k\right\rangle = 0.\end{aligned}$$

Consider the special case (Fubini-Study metric on $\mathbb{C}P^2$*) where*

$$\begin{aligned} a_i(r) &= \sin r \cos r \\ a_j &= a_k = \cos r.\end{aligned}$$

Perelman [**415**] showed that for metrics of the form

$$\begin{aligned} a_i(r) &= \frac{1}{10} r\left(1 + \phi(r)\sin(\log\log r)\right) \\ a_j(r) &= \frac{1}{10} r\left(1 + \phi(r)\sin(\log\log r)\right)^{-1} \\ a_k(r) &= \frac{1}{10} r\left(1 - \gamma(r)\right)\end{aligned}$$

where ϕ and γ are smooth functions satisfying

$\phi(r) = 0$ for $0 \leq r \leq \rho$, $\phi(r) > 0$ for $r > \rho$, $0 \leq \phi'(r) \leq r^{-2}$, $\left|\phi''(r)\right| \leq r^{-3}$,

and

$$\gamma(r) = 0 \text{ for } 0 \leq r \leq \rho/2,\ \gamma'(r) > 0$$

and

$$\gamma''(r) > 0 \text{ for } \rho/2 < r < \rho,\ \gamma'(r) = \left(r \log^{3/2} r\right)^{-1} \text{ for } r > \rho$$

(and smoothed out at the origin where $r = 0$), one has

$$|\mathrm{Rm}| = O\left(r^{-2}\right)$$
$$\mathrm{Rc}\left(\frac{\partial}{\partial r}, \frac{\partial}{\partial r}\right) \geq C / \left(r^2 \log^{3./2} r\right)$$
$$\mathrm{Rc}\left(f_i, f_i\right) \geq C/r^2.$$

In particular, the Ricci curvature of such metrics are positive. However, since $\phi(r) \geq c > 0$ for $r \geq \rho + 1$, and sine oscillates, we have the following:

THEOREM 7.29 (Perelman). *The asymptotic cones of these metrics are not unique.*

EXERCISE 7.30. *Prove the above results.*

6.1. Point picking.

THEOREM 7.31 (Infinite ASCR point picking). *Let* (M^n, g) *be a complete noncompact Riemannian manifold with* $\mathrm{ASCR}(g) = \infty$ *and let* $O \in M^n$. *There exists a sequence of points* $\{x_i\}_{i=1}^{\infty}$ *with* $d(x_i, O) \to \infty$ *and sequences* $\varepsilon_i \to 0$ *and* $r_i > 0$ *with* $R(x_i) r_i^2 \to \infty$ *such that the balls* $B(x_i, r_i)$ *are disjoint,* $d(x_i, O)/r_i \to \infty$, *and*

$$\sup_{B(x_i, r_i)} R \leq (1 + \varepsilon_i) R(x_i). \tag{7.17}$$

PROOF. Choose any sequences $\delta_i \to 0$ and $A_i \to \infty$ such that $A_i \delta_i^2 \to \infty$. Define $s_i > 0$ by

$$s_i \doteqdot \max\left\{s > 0 : \max_{B(O,s)} R(x) d(x, O)^2 \leq A_i\right\}. \tag{7.18}$$

Since $\mathrm{ASCR}(g) = \infty$, we have $s_i < \infty$ for each i. By the definition of s_i there exists $y_i \in B(O, s_i)$ such that $d(y_i, O) = s_i$ and

$$R(y_i) d(y_i, O)^2 = A_i. \tag{7.19}$$

Since $A_i \to \infty$, we have $s_i \to \infty$. Now choose $x_i \in M^n$ with $d(x_i, O) \geq s_i \to \infty$ so that

$$R(x_i) \geq (1 - \delta_i) \sup_{M^n - B(O, s_i)} R. \tag{7.20}$$

These will be the centers of the balls we will consider; their radii will be $r_i \doteqdot \delta_i s_i$ which satisfy $r_i / s_i \to 0$ so that $d(x_i, O)/r_i \to \infty$. Note that

$$R(x_i) \geq (1 - \delta_i) R(y_i) = (1 - \delta_i) A_i s_i^{-2}. \tag{7.21}$$

In particular, $R(x_i) r_i^2 \geq (1 - \delta_i) A_i \delta_i^2 \to \infty$. Since $d(x_i, O) \to \infty$ and $r_i / d(x_i, O) \to 0$, it is easy to see that after passing to a suitable subsequence, $B(x_i, r_i)$ are disjoint.

Finally we prove (7.17). Suppose $x \in B(x_i, r_i)$. If $d(x, O) \geq s_i$, then by (7.20)

$$R(x) \leq \frac{1}{1 - \delta_i} R(x_i).$$

On the other hand, if $d(x,O) < s_i$, then by (7.18), $R(x) \leq A_i d(x,O)^{-2}$. Now we use

$$d(x,O) \geq d(x_i,O) - d(x,x_i) \geq s_i - r_i = (1-\delta_i)\, s_i,$$

which implies

$$R(x) \leq A_i (1-\delta_i)^{-2} s_i^{-2} \leq (1-\delta_i)^{-3} R(x_i)$$

by (7.21). Since $(1-\delta_i)^{-3} \geq (1-\delta_i)^{-1}$, we choose $\varepsilon_i \doteqdot (1-\delta_i)^{-3} - 1$ and we have proved (7.17). □

6.2. Bumps of curvature. The geometries of ancient solutions are often product like at spatial infinity. By this we mean that given a fixed time, there often exists a sequence of points tending to spatial infinity whose corresponding rescalings limit to a solution which is the product of a line and a surface. The complementary case, which we want to rule out, is when the manifold has so-called curvature bumps tending to spatial infinity.

DEFINITION 7.32 (Curvature bump). *A ball $B(p,r)$ in a Riemannian manifold (M^n, g) is called a **curvature ε-bump** if*

$$\operatorname{sect}(g) \geq \frac{\varepsilon}{r^2} \qquad \text{in } B(p,r).$$

This notion is scale-invariant in the sense that if $B(p,r)$ is a curvature ε-bump with respect to a metric g, then $B\left(p, \sqrt{A}r\right)$ is a curvature ε-bump with respect to the metric Ag for any $A > 0$. We are interested in whether there can be lots of curvature bumps far away from each other.

DEFINITION 7.33. *Let $O \in M^n$. Given $\lambda > 0$ (usually large), a curvature ε-bump $B(p,r)$ is said to be **λ-remote** with respect to O if*

$$d(x,O) \geq \lambda r.$$

Given points p and q in a Riemannian manifold, we let pq denote a minimal geodesic from p to q. By the second variation of arc length formula one can see that long minimal geodesics tend to avoid positive curvature regions. A quantitative version of this fact is the following, whose proof relies on the repeated use of Toponogov's triangle comparison theorem.

PROPOSITION 7.34 (Repulsion principle). *For every $\varepsilon > 0$ and $n \geq 2$ there exists a $\lambda < \infty$ depending only on ε and n such that if (M^n, g) is a complete Riemannian manifold with nonnegative sectional curvature and if*

(1) *$x \in M^n$ is a point and $r > 0$ is a radius such that $\operatorname{sect}(g) \geq \frac{\varepsilon}{r^2}$ in $B(x, 3r)$,*
(2) *y_1 and y_2 are points with $\min\{d(y_1,x), d(y_2,x)\} \geq \lambda r$,*

then any minimal geodesic y_1y_2 joining y_1 and y_2 is disjoint from the ball of radius $s \doteqdot \min\{d(y_1,x), d(y_2,x)\}/\lambda \geq r$ centered at x ; that is, for every $w \in y_1y_2$, $d(w,x) \geq s$.

See §21 of [**267**] or [**143**] for the proof. The repulsion principle implies the following, which is the main result of this section.

THEOREM 7.35 (Finite number of curvature bumps). *For every $\varepsilon > 0$ and $n \geq 2$ there exists a $\lambda < \infty$ depending only on ε and n such that if (M^n, g) is a complete Riemannian manifold with nonnegative sectional curvature, then there are at most a finite number of disjoint λ-remote curvature ε-bumps.*

Proof. Given any $\varepsilon > 0$, let $\lambda < \infty$ be as in the proposition above. Suppose there are an infinite number of disjoint λ-remote curvature ε-bumps. Since any compact region can only have a finite number of disjoint curvature bumps, we can find an infinite sequence of disjoint λ-remote curvature ε-bumps $B(x_i, r_i)$ with $d(x_{i+1}, O) \geq 2d(x_i, O)$. Given any i and j with $i < j$, there exists a unique point w_{ij} on a minimal geodesic Ox_j with $d(w_{ij}, O) = d(x_i, O)$. By the repulsion principle, we have $d(w_{ij}, x_i) \geq \frac{1}{\lambda} d(x_i, O)$. Now we consider the triangle $x_i O w_{ij}$ and apply the Toponogov triangle comparison theorem and the law of cosines to obtain

$$\begin{aligned} d(w_{ij}, x_i)^2 &\leq d(x_i, O)^2 + d(w_{ij}, O)^2 - 2\cos(\measuredangle x_i O w_{ij})\, d(x_i, O)\, d(w_{ij}, O) \\ &= 2d(x_i, O)^2 (1 - \cos(\measuredangle x_i O x_j)). \end{aligned}$$

Hence $\cos(\measuredangle x_i O x_j) \leq 1 - \frac{1}{2\lambda^2}$, which implies $\measuredangle x_i O x_j \geq \theta$, for all i and j with $i < j$, and where $\theta = \cos^{-1}\left(1 - \frac{1}{2\lambda^2}\right) > 0$. This yields a contradiction.

6.3. Dimension reduction.

EXERCISE 7.36. *Show that if f is a C^∞ function with $\nabla_i \nabla_j f > 0$ everywhere and $\nabla f(O) = 0$ at some point $O \in M$ on a complete Riemannian manifold (M^n, g), then M^n is diffeomorphic to $\mathbb{R}^n$.*

HINT. This follows from Morse theory (see for example [**378**]).

THEOREM 7.37. *Let (M^n, g), where the dimension n is odd, be a complete steady gradient Ricci soliton on a noncompact manifold with positive sectional curvature and such that R attains its maximum at some point $O \in M$. Suppose there exists a sequence of points $\{x_i\}$ such that $d(x_i, O) \to \infty$, $\{r_i\}$ is a sequence of radii with $r_i^2 R(x_i) \to \infty$ and*

$$R(x) \leq CR(x_i) \text{ for all } x \in B(x_i, r_i)$$

for some constant $C < \infty$. (We can obtain such a sequence from the point picking Theorem 7.31.) In addition, assume that $\{x_i\}$ can be chosen with the above properties and

$$R(x_i) \geq c \max_{\Sigma_{\phi(x_i)}} R$$

where $R_{ij} = \nabla_i \nabla_j \phi$ and $c > 0$ is independent of i. Then there exists a constant $c' > 0$ such that

$$\operatorname{inj}(x_i) \geq c' R(x_i)^{-1/2}.$$

PROOF. The steady gradient Ricci soliton equation is

$$\nabla_i \nabla_j \phi = R_{ij} > 0. \tag{7.22}$$

Recall from taking the divergence of (7.22) we have

$$\nabla R + 2\,\mathrm{Rc}\,(\nabla \phi) = 0 \tag{7.23}$$

so that $|\nabla \phi|\,(O) = 0$ since $|\nabla R|\,(O) = 0$ and $\mathrm{Rc} > 0$. Hence ϕ attains its minimum at O. Substituting $\nabla\nabla\phi$ for Rc in (7.23) implies

$$R + |\nabla \phi|^2 = \mathrm{const} = R\,(O)$$

since $|\nabla \phi|\,(O) = 0$. Thus $|\nabla \phi| \leq \sqrt{R\,(O)}$ on M. Let γ be a unit speed geodesic emanating from O. Then $\frac{d}{ds}\phi\,(\gamma\,(s)) \leq |\nabla \phi| \leq \sqrt{R\,(O)}$ so that, normalizing ϕ so that $\phi\,(O) = 0$, we have

$$\phi\,(x) \leq \sqrt{R\,(O)}d\,(x, O)\,.$$

To get a lower bound for $\frac{d}{ds}\phi\,(\gamma\,(s)) = \nabla \phi \cdot \dot{\gamma}$ we argue as follows:

$$\frac{d}{ds}\langle \nabla \phi, \dot{\gamma} \rangle = \nabla\nabla\phi\,(\dot{\gamma}, \dot{\gamma}) = \mathrm{Rc}\,(\dot{\gamma}, \dot{\gamma}) > 0$$

so that

$$\langle \nabla \phi\,(\gamma\,(s))\,, \dot{\gamma}\,(s) \rangle \geq \langle \nabla \phi\,(\gamma\,(\varepsilon))\,, \dot{\gamma}\,(\varepsilon) \rangle\,.$$

It is not hard to see that there exists $\varepsilon > 0$ and $A > 0$ such that $\langle \nabla \phi, \dot{\gamma}\,(\varepsilon) \rangle \geq A$ for any unit speed geodesic γ emanating from O (see Remark 7.38 below). We then have $\frac{d}{ds}\phi\,(\gamma\,(s)) \geq A$ for all $s \geq \varepsilon$ and integrating this yields

$$\phi\,(x) \geq Ad\,(x, O)\,.$$

Consider the level sets of ϕ

$$\Sigma_c \doteqdot \phi^{-1}\,(c)\,.$$

By the above estimates, for every $x \in \Sigma_c$ where c is sufficiently large

$$\frac{c}{\sqrt{R\,(O)}} \leq d\,(x, O) \leq \frac{c}{A}. \tag{7.24}$$

Let II denote the second fundamental form of Σ_c recall from (**??**):

$$\mathrm{II}\,(X, Y) = \frac{(\nabla\nabla\phi)\,(X, Y)}{|\nabla \phi|} = \frac{\mathrm{Rc}\,(X, Y)}{|\nabla \phi|} > 0. \tag{7.25}$$

From (**??**) we have for any $c_0 > \phi\,(O)$, there exists $b_0 > 0$ such that $|\nabla \phi| \geq b_0$ on Σ_c for $c \geq c_0$. Hence, by (7.25) and the bound on the Ricci tensor ($\mathrm{Rc} > 0$ and the scalar curvature achieves its maximum), for some $C_1 < \infty$

$$|\mathrm{II}\,(x)| \leq C_1 R\,(x) \;\text{ for } x \in \Sigma_c$$

for c sufficiently large.

Now by hypothesis we have a sequence of points $\{x_i\}$ such that $d\,(x_i, O) \to \infty$, $\{r_i\}$ is a sequence of radii with $r_i^2 R\,(x_i) \to \infty$ and

$$R\,(x) \leq C_2 R\,(x_i) \;\text{ for all } x \in B\,(x_i, r_i)$$

for some constant $C_2 < \infty$, and

$$R(x_i) \geq c_3 \max_{\Sigma_{\phi(x_i)}} R$$

where $c_3 > 0$ is independent of i. Since

$$\nabla R \cdot \nabla \phi = -2 \operatorname{Rc}(\nabla \phi, \nabla \phi) \leq 0$$

we have R is decreasing along the integral curves to $\nabla \phi$. In particular,

$$R(x) \leq \frac{1}{c_3} R(x_i) \text{ for all } x \in \Sigma_c \text{ where } c \geq \phi(x_i).$$

Then

$$|\mathrm{II}(x)| \leq \frac{C_1}{c_3} R(x_i) \text{ for } x \in \Sigma_c \text{ where } c \geq \phi(x_i).$$

Since $\operatorname{sect}(M) > 0$ and $\mathrm{II} > 0$, by the Gauss equations, for any linearly independent $X, Y \in T\Sigma_c$

$$(\operatorname{Rm}_{\Sigma_c})(X,Y,Y,X) = (\operatorname{Rm}_M)(X,Y,Y,X) + \mathrm{II}(X,X)\,\mathrm{II}(Y,Y) - [\mathrm{II}(X,Y)]^2 > 0.$$

Furthermore, from this we have

$$|\operatorname{Rm}_{\Sigma_c}| \leq C_4 R(x_i) \text{ for } x \in \Sigma_c \text{ where } c \geq \phi(x_i).$$

Since M^n is odd-dimensional, Σ_c is even-dimensional and we can apply the Klingenberg injectivity radius estimate to obtain

$$\operatorname{inj}(\Sigma_c) \geq \frac{c_5}{\sqrt{R(x_i)}}$$

for some $c_5 > 0$ independent of i. Hence there exists $c_6 > 0$ independent of i such that

$$\operatorname{Vol}_{\Sigma_c}\left(B_{\Sigma_c}\left(\gamma(c), \frac{r}{\sqrt{R(x_i)}}\right)\right) \geq c_6 \left[r R(x_i)^{-1/2}\right]^{n-1}$$

for any $r \leq c_5$. Let $\gamma : \left[\phi(x_i), \phi(x_i) + R(x_i)^{-1/2}\right] \to M$ be the integral curve to $\nabla \phi$ with $\gamma(\phi(x_i)) = x_i$. Since $|\nabla \phi| \leq \sqrt{R(O)}$, by the co-area formula (Lemma **??**) we have

$$\begin{aligned}
\operatorname{Vol}\left(B\left(x_i, R(x_i)^{-1/2}\right)\right) &\geq \int_{\phi(x_i)}^{\phi(x_i)+\frac{1}{2\sqrt{R(O)}} R(x_i)^{-1/2}} \left(\int_{B_{\Sigma_c}\left(\gamma(c), \frac{c_7}{\sqrt{R(x_i)}}\right)} \frac{1}{|\nabla \phi|} d\sigma\right) dc \\
&\geq \frac{1}{\sqrt{R(O)}} c_6 \left[c_7 R(x_i)^{-1/2}\right]^{n-1} \frac{1}{2\sqrt{R(O)}} R(x_i)^{-1/2} \\
&= \frac{c_6 c_7^{n-1}}{2R(O)} \left[R(x_i)^{-1/2}\right]^n
\end{aligned}$$

where $c_7 \doteqdot \min\left\{c_5, \frac{1}{2}\right\}$. This implies the desired injectivity estimate at $\{x_i\}$. $\square$

REMARK 7.38. *For any unit speed geodesic* $\gamma : [0, L] \to M$ *emanating from* O *we have*

$$\frac{d^2}{ds^2}\phi(\gamma(s)) = (\nabla\nabla\phi)(\dot{\gamma}, \dot{\gamma}) > 0 \tag{7.26}$$

and $\left.\frac{d}{ds}\right|_{s=0}\phi(\gamma(s)) = 0$, *so that for any* $\varepsilon > 0$ *there exists* $c > 0$ *(independent of* γ*) such that*

$$\frac{d}{ds}\phi(\gamma(s)) \geq c$$

for $s \geq \varepsilon$. *The way to see this is to first choose any* $\varepsilon < \operatorname{inj}(O)$ *and to note that*

$$\inf_{\gamma(0)=O} \left.\frac{d}{ds}\right|_{s=\varepsilon} \phi(\gamma(s)) \doteqdot c > 0$$

(the LHS *is the same as* $\inf_{d(x,O)=\varepsilon} \frac{\partial\phi}{\partial r}(x)$*), and then to apply (7.26) for larger* ε.

7. Hamilton's partial classification of 3-dimensional singularities

By the Hamilton-Ivey curvature estimate, singularity models have nonnegative sectional curvature. The relative topological and geometric simplicity of such solutions is one of the reasons their classification is possible.

The following partial classification result for 3-dimensional Type I singularities was proved by Hamilton, §24 of [**267**].

THEOREM 7.39 (3-dimensional Type I singularity models). *If a solution* $(M^3, g(t))$, $t \in [0, T)$, *to the Ricci flow on a closed* 3*-manifold develops a Type I singularity, then either:*

(1) (Spherical space form) M^3 *is diffeomorphic to a spherical space form and the solution* $\tilde{g}(t)$ *to the normalized Ricci flow with* $\tilde{g}(0) = g(0)$ *converges to a constant positive sectional curvature metric, or*
(2) (Exist neck-like points) *there exists a sequence of points and times* (x_i, t_i) *with* $t_i \to T$ *such that the corresponding dilated solutions* $(M^3, g_i(t))$ *defined by (7.1) converge to a quotient of the shrinking round cylinder* $S^2 \times \mathbb{R}$.

PROOF. (*Sketch.* See see Theorem 24.7 of [**267**] or section 9.4 of [**153**] for more details.) By (3.25), there exists a constant ρ such that $R+\rho \geq c > 0$ at all points and times. Given $\varepsilon > 0$, consider the quantity

$$F \doteqdot (T-t)^{\varepsilon} \frac{\left|\operatorname{Rm} - \frac{1}{3}RI_{\wedge^2}\right|^2}{(R+\rho)^{2-\varepsilon}}.$$

Note that $\operatorname{Rm} - \frac{1}{3}RI_{\wedge^2}$ is the trace-free part of Rm. Certainly we cannot show that in general $F \to 0$. The idea is to show that if case 2 does not hold, that is, there does not exist a sequence of neck-like points, then

$\lim_{t\to T}\max_{x\in M^3} F(x,t) = 0$. In particular, suppose that there exist $c, \delta > 0$ such that at every point and time, either

$$(T-t)\,|\mathrm{Rm}| < c$$

or

$$\frac{|\mathrm{Rm} - R\theta\otimes\theta|}{|\mathrm{Rm}|} > \delta$$

for every unit 2-form θ. One can show that there exists a constant $a > 0$ such that

$$\frac{\partial}{\partial t}F \le \Delta F + \frac{2(1-\varepsilon)}{R+\rho}\langle\nabla R, \nabla F\rangle - \frac{a}{T-t}F.$$

Applying the maximum principle, one deduces that

$$\lim_{t\to T}\max_{x\in M^3} F(x,t) = 0.$$

With a little more work, one can actually show that the solution $\tilde{g}(t)$ to the normalized flow with $\tilde{g}(0) = g(0)$ converges to a constant positive sectional curvature metric on M^3. □

The intuition underlying the statement of the above theorem is that a Type I singularity on a 3-manifold should either represent convergence of the normalized flow to a spherical space form (case 1) or the formation of a neck pinch (what is expected to happen in case 2).

Recall that given a solution $(M^n, g(t))$ and a sequence of points and times (x_i, t_i), we considered the dilated solutions $(M^n, g_i(t))$, where

$$g_i(t) = K_i g\left(t_i + K_i^{-1}t\right), \quad K_i \doteqdot |\mathrm{Rm}(x_i, t_i)|.$$

Given a Type I singularity, for *any* sequence of points and times (x_i, t_i) with $(T-t_i)\,|\mathrm{Rm}(x_i,t_i)| \ge c$ for some $c > 0$, the dilated solutions $(M^n, g_i(t))$ satisfy a uniform curvature bound. After passing to a subsequence, these solutions $(M^n, g_i(t), x_i)$ converge to a complete ancient Type I solution $(M_\infty^n, g_\infty(t), x_\infty)$. When $n = 3$, by the Hamilton-Ivey estimate, we know that $g_\infty(t)$ has nonnegative sectional curvature. Moreover, by Theorem 7.39, we have that either $(M_\infty^n, g_\infty(t))$ is isometric to a shrinking spherical space form or there exists another sequence of points and times $(\bar{x}_i, \bar{t}_i)$ whose dilated solutions $(M^n, \bar{g}_i(t), \bar{x}_i)$ converge to a shrinking round cylinder $S^2\times\mathbb{R}$; these two possibilities are mutually exclusive.

For Type II singularities, to obtain a partial classification, we need to invoke Perelman's no local collapsing theorem, which provides us with the injectivity estimate that we previously unknown. With this aid, which also rules out the cigar soliton as a singularity model, Hamilton's result becomes strengthened to the following.

THEOREM 7.40 (Partial classification of Type IIa singularity models). *If a solution* $\left(M^3, g(t)\right)$, $t\in[0,T)$, *to the Ricci flow on a closed* 3*-manifold develops a Type IIa singularity, then*

(1) *there exists a sequence of points and times* (x_i, t_i) *with* $t_i \to T$ *such that the corresponding dilated solutions* $\left(M^3, g_i(t)\right)$ *defined by (7.1) converge to a steady Ricci soliton with positive sectional curvature.*
(2) *In addition, there exist points and times* $(\bar{x}_i, \bar{t}_i)$ *with* $\bar{t}_i \to T$ *such that the corresponding dilated solutions* $\left(M^3, \bar{g}_i(t)\right)$ *converge to the shrinking round cylinder* $S^2 \times \mathbb{R}$.

PROOF. For a Type IIa singularity, motivated by the eternal solutions application of the matrix Harnack estimate (Theorem 8.36), it is useful to be more careful in how we pick the sequence (x_i, t_i). We restrict ourselves to the 3-dimensional case, where we have the estimate $C\,|\mathrm{Rm}| - C \leq R \leq |\mathrm{Rm}|$. To start, choose any sequence of times $T_i \to T < \infty$. Let

$$\Omega_i \doteqdot \sup_{M^3 \times [0,T_i]} (T_i - t)\,|R(x,t)|$$

which satisfies $\Omega_i \to \infty$. One can choose points and times (x_i, t_i) such that

$$\lim_{i \to \infty} \Omega_i^{-1} (T_i - t_i)\,|R(x_i, t_i)| = 1.$$

In this case, the dilated solutions $\left(M^3, g_i(t), x_i\right)$ converge (recall Perelman's no local collapsing Theorem **??** provides the required injectivity radius estimate) to a complete solution $\left(M^3_\infty, g_\infty(t), x_\infty\right)$ to the Ricci flow defined on the eternal time interval $(-\infty, \infty)$ with the property that $\mathrm{Rm}(g_\infty(t)) \geq 0$ and $R(x_\infty, 0) = \sup_{M^3_\infty \times (-\infty,\infty)} R > 0$. By Theorem 8.36, we have that $\left(M^3_\infty, g_\infty(t)\right)$ is a steady gradient Ricci soliton. The strong maximum principle (see section 4 of Chapter 4) implies that either $g_\infty(t)$ has positive sectional curvature or the universal covering solution $\tilde{g}_\infty(t)$ splits as the product of a steady gradient Ricci soliton on a surface with $\mathbb{R}$. The latter case is not possible since by Lemma 4.7 the surface Ricci soliton must be the cigar, which is ruled out by the no local collapsing theorem. Hence $g_\infty(t)$ must have positive sectional curvature and dimension reduce to a round cylinder $S^2 \times \mathbb{R}$. □

The above result is in agreement with the degenerate neck pinch picture (see section 3).

EXERCISE 7.41 (Chu). *Show that if* g *evolves by Ricci flow and* α *is a 1-form evolving by*

$$\frac{\partial}{\partial t}\alpha = \Delta_d \alpha = \Delta\alpha - \mathrm{Rc}(\alpha),$$

then

$$\frac{\partial}{\partial t}|\alpha|_g^2 = \Delta|\alpha|_g^2 - 2|\nabla\alpha|^2. \tag{7.27}$$

Hence, by the maximum principle, we have

$$\max_{M^n} |\alpha(t)|_{g(t)}^2$$

is a nonincreasing function of t.

Equation (7.27) has the following application to the Ricci flow due to Ilmanen and Knopf [**296**].

THEOREM 7.42. *If $(M^n, g(t))$ is a solution to the Ricci flow on a closed manifold and if $\omega \in H^1(M^n, \mathbb{Z})$ is an element of infinite order, then there exists $c = c(g(0), \omega) > 0$ such that the length with respect to $g(t)$ of any curve representing ω is bounded below by c for all $t \geq 0$.*

This result has the following consequence.

COROLLARY 7.43. *No compact quotient of $S^2 \times \mathbb{R}$ can occur as the limit of dilations about a singularity for a solution of the Ricci flow on a closed 3-manifold.*

REMARK 7.44. *As we shall see in Volume 2, this result is also an easy consequence of Perelman's no local collapsing theorem.*

8. Some conjectures about ancient solutions

Recall that a solution $(M^n, g(t))$ to the Ricci flow is called ancient if it is defined on an interval of the form $(-\infty, \omega)$, where $\omega \in \mathbb{R}^+ \cup \{\infty\}$, and we say that an ancient solution is Type I if $\sup_{M^n \times (-\infty,0]} |t| |\mathrm{Rm}(x,t)| < \infty$.

In dimension 2, it may be hoped that there is a nice classification of ancient solutions.

PROBLEM 7.45 (Classification of 2-d ancient solutions). *Suppose that $\left(M^2, g(t)\right)$ is a complete ancient solution with bounded curvature (the bound may depend on time). By Lemma 7.4 we have either $R[g(t)] \equiv 0$ or $R[g(t)] > 0$ everywhere. In the latter case is $\left(M^2, g(t)\right)$ necessarily homothetic to one of the following solutions?*

(1) *The cigar soliton.*
(2) *The constant curvature shrinking S^2 or $\mathbb{R}P^2$.*
(3) *The Rosenau solution or its $\mathbb{Z}_2$ quotient.*

Note that we have not assumed that $\left(M^2, g(t)\right)$ is an ancient κ-solution. A rather naive reason for believing the above 3 cases may be the only possibilities is the following. By Propositions 7.18 and 7.22, if $\left(M^2, g(t)\right)$ is not isometric to a constant curvature shrinking S^2 or $\mathbb{R}P^2$, then there exists a backward limit (dilate about an appropriate sequence of points and times (x_i, t_i) with $t_i \to -\infty$) which is isometric to a cigar soliton. We expect that the noncompact case should correspond to the cigar soliton and compact case to the Rosenau solution.

In dimension 3, based on the analogous reasoning, one may hope for a similar classification for ancient κ-solutions. Recall that an ancient solution is an ancient κ-solution if it is complete, nonflat with bounded nonnegative curvature operator and κ-noncollapsed on all scales for some $\kappa > 0$. The following is mostly in [**417**].

PROBLEM 7.46 (Classification of 3-d κ-ancient solutions). *Suppose that* $(M^3, g(t))$ *is an ancient* κ*-solution for some* $\kappa > 0$ *with positive sectional curvature. Is* $(M^3, g(t))$ *necessarily homothetic to one of the following solutions?*

(1) *The Bryant soliton.*
(2) *A shrinking spherical space form.*
(3) *A rotationally symmetric solution on* S^3 *or its* $\mathbb{Z}_2$ *quotient (by the antipodal map) on* $\mathbb{R}P^3$*. Is such a solution unique up to homothety?*

Here, unless we are on a shrinking spherical space form, we can show that a backward limit is a steady Ricci soliton with positive sectional curvature. The dimension reduction must be a round 2-sphere since the cigar is not κ-noncollapsed for any $\kappa > 0$. Since the Ricci soliton is essentially asymptotically rotationally symmetric, it is reasonable to conjecture that it must in fact be globally rotationally symmetric. Now the noncompact case should correspond to the Bryant soliton and compact case to a rotationally symmetric solution on S^3 or its $\mathbb{Z}_2$ quotient on $\mathbb{R}P^3$ (analogous to the Rosenau solution).

PROBLEM 7.47 (Removing the κ condition). *In the previous problem, if we remove the assumption of being* κ*-noncollapsed for some* $\kappa > 0$*, then we may ask if the only other possibilities are the quotients of the product of an ancient 2-dimensional solution with* $\mathbb{R}$*. At the moment it is not clear whether or not there is a complete Ricci soliton with positive sectional curvature on a noncompact* 3*-manifold (diffeomorphic to* $\mathbb{R}^3$*) which dimension reduces to the cigar soliton. This question (and the analogous one for the mean curvature flow) was posed by Altschuler, Grayson and Hamilton.*[2]

In all dimensions, one can ask the following, which is mostly in [**267**].

PROBLEM 7.48 (Ancient solutions with positive curvature operator). *Suppose that* $(M^n, g(t))$, $t \in (-\infty, \omega)$, *where* $\omega > 0$, *is a complete ancient solution to the Ricci flow with positive curvature operator with*

$$\sup_{M \times (-\infty, 0]} |t| \, |\mathrm{Rm}(x,t)| < \infty$$

(Type I). Is M^n *closed? Is* $(M^n, g(t))$ *in fact a shrinking spherical space form?*

In section 22 of [**267**] Hamilton wrote:

> We do not know any examples of complete noncompact ancient solutions of positive curvature operator with $Rs^2 < \infty$ and $R|t| < \infty$, and we conjecture none exist, since the curvature has had plenty of space and time to dissipate.

[2]Apparently each of the three attributes this question to the other two! In the absence of a consensus we attribute it to all three. :-)

More explicitly, by $Rs^2 < \infty$ and $R|t| < \infty$, Hamilton meant finite asymptotic scalar curvature ratio:

$$\mathrm{ASCR}(g(t)) < \infty, \quad \text{and} \quad \sup_{M \times (-\infty,0]} |t| |\mathrm{Rm}(x,t)| < \infty.$$

The follow results, which are inspired or essentially follow from the ideas in [**267**] and [**270**], were proved in [**156**].

PROPOSITION 7.49 (Chow-Lu - 3-d ancient Type I solutions). *Suppose that $(M^3, g(t))$ is a complete ancient solution of the Ricci flow with bounded positive sectional curvature and*

$$\sup_{M^3 \times (-\infty,0]} |t|^\gamma R(x,t) < \infty \tag{7.28}$$

for some $\gamma > 0$. Then either

(1) *$(M^3, g(t))$ is isometric to a shrinking spherical space form, or*
(2) *there exists a constant $c > 0$ such that for all $\tau \in (-\infty, 0]$ and all $\delta > 0$, there exist a point and time (x_0, t_0) with $t_0 < \tau$ such that*

$$|\mathrm{Rm}(x_0,t_0)| \cdot |t_0| \geq c > 0$$

and there exists a unit 2-form θ such that

$$|\mathrm{Rm} - R(\theta \otimes \theta)| \leq \delta |\mathrm{Rm}|$$

at (x_0, t_0).

By applying the compactness theorem and the fact that one has an injectivity radius estimate for complete noncompact manifolds with bounded positive sectional curvature, we have the following.

COROLLARY 7.50. *If $(M^3, g(t))$, $t \in (-\infty, \omega)$, is a complete noncompact Type I ancient solution of the Ricci flow with bounded positive sectional curvature on an orientable 3-manifold, then there exists a sequence of points and times $(x_i, t_i) \in M^3 \times (-\infty, \omega)$ such that the dilated and translated solutions $(M^3, g_i(t), x_i)$, $t \in (-\infty, \omega_i)$, where*

$$g_i(t) = R(x_i,t_i) \cdot g\left(t_i + \frac{t}{R(x_i,t_i)}\right)$$

and $\omega_i = R(x_i,t_i)(T - t_i)$, limit to a solution $(M^3_\infty, g_\infty(t), x_\infty)$, $t \in (-\infty, \omega_\infty)$, to the Ricci flow isometric to the standard shrinking cylinder $S^2 \times \mathbb{R}$.

A simple consequence of this is:

COROLLARY 7.51 (3-d gradient shrinking solitons). *A gradient shrinking soliton on a closed 3-manifold with positive sectional curvature is isometric to a shrinking constant positive sectional curvature metric.*

With more work, one can prove the following. However in Volume 2 we shall see a generalization of this result by Perelman (see the remarks before Proposition 4.19.)

PROPOSITION 7.52 (Chow-Lu). *If* $\left(M^3, g(t)\right)$ *is a complete noncompact Type I ancient solution to the Ricci flow on an orientable* 3*-manifold with positive sectional curvature, then* $\operatorname{ASCR}(g(t)) = \infty$ *for all* t.

In addition, if we assume our 3-dimensional ancient solution is non-flat, Type I and κ-noncollapsed, then M^n is closed.

PROBLEM 7.53. *Are Type I ancient solutions* κ*-solutions?*

More ambitiously, one can ask:

PROBLEM 7.54. *Is it true that under hypothesis (7.28) in Proposition 7.49 that case (2) cannot occur?*

9. Notes and commentary

§1. The idea of dilating about a singularity has a long history in geometric analysis and partial differential equations. For semilinear heat equations, there are the works of Escobedo-Kavian [**200**], Giga-Kohn [**226**], and many others.

§3. For the mean curvature flow of a surface in euclidean 3-space, Angenent and Velazquez [**25**] have proved the existence of a degenerate neck pinch. Numerical studies of a degenerate neck pinch have been conducted by Garfinkle and Isenberg [**221**].

CHAPTER 8

Harnack type estimates

Let $(M^n, g(t))$ be a solution to the Ricci flow on a time interval $\mathcal{I} \subset \mathbb{R}$. Recall that one of the estimates used in understanding the long time behavior of the solution is the **differential Harnack estimate**. In this chapter we shall discuss various Harnack type estimates in the form of gradient estimates. Since the Harnack estimates for Ricci flow are similar in spirit to the original Li-Yau estimate for positive solutions of the heat equation, we begin by presenting the latter. Next, for simplicity we consider the case of the Ricci flow on surfaces. Motivated by gradient Ricci solitons, we discuss Hamilton's matrix Harnack estimate for solutions with nonnegative curvature operator of the Ricci flow in any dimension. The trace version of this estimate generalizes to solutions of the linearized Ricci flow system. Finally we consider a pinching estimate for solutions of the linearized Ricci flow system in dimension 3.

1. Li-Yau estimate for the heat equation

Recall the heat kernel on $\mathbb{R}^n$ is given by

$$h(x,t) = (4\pi t)^{-n/2} \exp\left\{-\frac{|x|^2}{4t}\right\}.$$

This solution tends to the delta function centered at the origin as $t \to 0$. Note that h is **self-similar** (Ricci solitons are the analogues of this):

$$h(cx, c^2 t) = c^{-n} h(x,t)$$

for any positive constant c. Taking the logarithm of h, we have

$$\log h = -\frac{n}{2}\log(4\pi t) - \frac{|x|^2}{4t},$$

with Laplacian

$$\Delta \log h + \frac{n}{2t} = 0. \tag{8.1}$$

It is rather surprising that one can obtain the following sharp estimate.

Theorem 8.1 (Li and Yau 1986 - Differential Harnack estimate, [**346**]). *If $u : M^n \times [0, \infty) \to \mathbb{R}$ is a positive solution to the heat equation on a*

Riemannian manifold (M^n, g) *with nonnegative Ricci curvature, then*

$$\boxed{\frac{\partial}{\partial t}\log u - |\nabla \log u|^2 + \frac{n}{2t} = \Delta \log u + \frac{n}{2t} \geq 0.} \tag{8.2}$$

REMARK 8.2. *The estimate is sharp in the sense that (8.1) implies equality holds in the case of the fundamental solution on euclidean space.*

PROOF. (*Idea.*) We calculate using (1.60) that

$$\begin{aligned}\frac{\partial}{\partial t}(\Delta \log u) &= \Delta\left(\frac{\partial}{\partial t}\log u\right) = \Delta\left(\Delta \log u + |\nabla \log u|^2\right)\\ &= \Delta(\Delta \log u) + 2\nabla \log u \cdot \Delta \nabla \log u + 2|\nabla\nabla \log u|^2\\ &= \Delta(\Delta \log u) + 2\nabla \log u \cdot \nabla \Delta \log u + 2|\nabla\nabla \log u|^2\\ &+ 2R_{ij}\nabla_i \log u \nabla_j \log u\\ &\geq \Delta(\Delta \log u) + 2\nabla \log u \cdot \nabla(\Delta \log u) + \frac{2}{n}(\Delta \log u)^2,\end{aligned}$$

since $\mathrm{Rc} \geq 0$. On a closed manifold the result now follows from the maximum principle since the solution to the ODE $\frac{dq}{dt} = \frac{2}{n}q^2$ with $\lim_{t\to 0} q(t) = -\infty$ is $q(t) = -\frac{n}{2t}$.

In the complete case, one applies the maximum principle to the quantity $F = \varphi\cdot\left(\Delta \log u + \varepsilon |\nabla \log u|^2\right)$, where $\varepsilon > 0$ and φ is a radial cutoff function (with respect to some choice of origin). See [**346**] or [**447**], §4.1 for details. □

Integrating this inequality over graphs $(\gamma(t), t)$ of paths $\gamma : [t_1, t_2] \to M^2$ joining points x_1 and x_2, we have a classical type Harnack inequality.

COROLLARY 8.3 (Comparing the solution at different points and times). *Suppose* $\mathrm{Rc}(g) \geq 0$. *For any* $x_1, x_2 \in M^n$ *and* $0 < t_1 < t_2$,

$$\boxed{\frac{u(x_2,t_2)}{u(x_1,t_1)} \geq \left(\frac{t_2}{t_1}\right)^{-\frac{n}{2}} \exp\left(-\frac{1}{4}\frac{d(x_1,x_2)^2}{t_2 - t_1}\right).} \tag{8.3}$$

PROOF. Indeed, for such a path γ, we have

$$\begin{aligned}\log\frac{u(x_2,t_2)}{u(x_1,t_1)} &= \int_{t_1}^{t_2} \frac{d}{dt}[\log u(\gamma(t), t)]\,dt\\ &= \int_{t_1}^{t_2}\left(\frac{\partial}{\partial t}\log u + \nabla \log u \cdot \frac{d\gamma}{dt}\right)dt\\ &\geq \int_{t_1}^{t_2}\left(|\nabla \log u|^2 - \frac{n}{2t} + \nabla \log u \cdot \frac{d\gamma}{dt}\right)dt\\ &\geq -\frac{n}{2}\log\left(\frac{t_2}{t_1}\right) - \frac{1}{4}\int_{t_1}^{t_2}\left|\frac{d\gamma}{dt}(t)\right|^2 dt\end{aligned}$$

and the result follows from taking γ to be a constant speed minimal geodesic (so that $\left|\frac{d\gamma}{dt}(t)\right| \equiv \frac{d(x_1,x_2)}{t_2-t_1}$). □

REMARK 8.4. *Note that the exponent* $\frac{1}{4}\frac{d(x_1,x_2)^2}{t_2-t_1}$ *in (8.3) is the analogue of Perelman's* ℓ*-function defined in Volume 2. We may think of* $\frac{1}{4}\frac{d(x_1,x_2)^2}{t_2-t_1}$ *as a space-time distance between* (x_1,t_1) *and* (x_2,t_2).

In [**346**], estimates are also obtained in the case where there is a negative lower bound for the Ricci curvature of (M^n,g) as well as the case where there is a potential function added to the heat equation so that u is a positive solution of $\frac{\partial u}{\partial t} = \Delta u + V \cdot u$, where $V : M^n \to \mathbb{R}$.

REMARK 8.5. *If we define* $f(x,t) \doteqdot -\log\left((4\pi t)^{n/2} u(x,t)\right)$ *so that* $u = (4\pi t)^{-n/2} e^{-f}$, *then* f *satisfies*

$$\frac{\partial f}{\partial t} = \Delta f - |\nabla f|^2 - \frac{n}{2t} \tag{8.4}$$

and we may rewrite (8.2) as

$$\frac{\partial f}{\partial t} + |\nabla f|^2 = \Delta f - \frac{n}{2t} \leq 0 \tag{8.5}$$

and (8.3) as

$$f(x_2,t_2) - f(x_1,t_1) \leq \frac{d(x_1,x_2)^2}{4(t_2-t_1)}. \tag{8.6}$$

Now suppose u *is a fundamental solution of the heat equation centered at some point* $p \in M^n$ *so that* $\lim_{t\to 0} u = \delta_p$. *Then* $\lim_{t\to 0} f(p,t) = 0$ *and we have*

$$f(x,t) \leq \frac{d(x,p)^2}{4t}. \tag{8.7}$$

We phrase the Harnack inequality this way in view of the ℓ*-function for the Ricci flow which we consider in Volume 2. In terms of the fundamental solution, this says*

$$u(x,t) \geq (4\pi t)^{-n/2} e^{-\frac{d(x,p)^2}{4t}}. \tag{8.8}$$

REMARK 8.6. *The Li-Yau estimate* $\Delta \log u + \frac{n}{2t} \geq 0$ *for positive solutions of the heat equation on manifolds with* $\mathrm{Rc} \geq 0$ *is a generalized Laplacian comparison theorem. This because taking* u *to be the fundamental solution and* $t \to 0$, *it implies the inequality (see (1.70))*

$$\Delta\left(d_p^2\right) - 2n \leq 0,$$

where d_p *is the distance to a point* p, *especially when* u *is a fundamental solution centered at* p.

The precursor to the parabolic Li-Yau gradient estimate is the following **Liouville type theorem**.

THEOREM 8.7 (Yau 1975, [**526**]). *If (M^n, g) is complete Riemannian manifold with nonnegative Ricci curvature and u is a positive harmonic function, then u is constant.*

The proof is based on the following elliptic **gradient estimate**: given $p \in M^n$, there exists $C < \infty$ depending only on n such that

$$\sup_{x \in B(p,r)} |\nabla \log u|(x) \leq \frac{C}{r} \tag{8.9}$$

for all $r < \infty$. The Liouville theorem follows immediately from taking $r \to \infty$. The idea of the proof of (8.9) is to consider the function

$$F(x) \doteqdot \left(r^2 - d(x,p)^2\right) |\nabla \log u|(x)$$

and by taking its Laplacian and applying the maximum principle (and using the $\mathrm{Rc} \geq 0$ hypothesis) to derive the inequality:

$$F^2 - C_1 F - C_2 r^2 \leq 0, \tag{8.10}$$

where C_1 and C_2 depend only on n. This implies $F \leq C_3 r$ for some C_3 and hence

$$|\nabla \log u|(x) \leq \frac{C_3}{r - d(x,p)} \tag{8.11}$$

for all $x \in B(p,r)$. The estimate (8.9) now follows from restricting this estimate to $x \in B(p, r/2)$. Here we have swept under the rug one technical point: the distance function $d(\cdot, p)$ is not smooth at cut points of p. Implicitly, in the above proof, we have assumed that the point x_0 where F attains its maximum is not a cut point of p. If x_0 is a cut point of p, we use **Calabi's trick** and argue as follows.

Let $\gamma : [0, L] \to M^n$, where $L \doteqdot d(x,p)$, be a minimal geodesic joining p to x_0. Let $p_\varepsilon \doteqdot \gamma(\varepsilon)$ for $\varepsilon > 0$. Now define

$$F_\varepsilon(x) \doteqdot \left(r^2 - (d(x, p_\varepsilon) + \varepsilon)^2\right) |\nabla \log u|(x).$$

Since, by the triangle inequality, $d(x, p_\varepsilon) + \varepsilon \geq d(x,p)$ and $d(x_0, p_\varepsilon) + \varepsilon = d(x_0, p)$, we have

$$F_\varepsilon(x) \leq F(x) \text{ for all } x \in M^n$$
$$F_\varepsilon(x_0) = F(x_0).$$

So clearly F_ε attains its maximum at x_0. Since the function $x \mapsto d(x, p_\varepsilon)$ is smooth in a neighborhood of x_0, we may apply the maximum principle to obtain an estimate for $F_\varepsilon(x_0)$ which also depends on ε. Taking the limit as $\varepsilon \to 0$ we obtain the estimate $F(x_0) = \lim_{\varepsilon \to 0} F_\varepsilon(x_0) \leq C_3 r$ since the estimate for F_ε tends to the original estimate for F as $\varepsilon \to 0$.

EXERCISE 8.8. *Work out an inequality for F_ε similar to (8.10) and show that the resulting estimate for F_ε limits to the desired estimate (8.11).*

Yau's gradient estimate implies the following Harnack inequality (we state it for the more general case of Ricci curvature bounded from below). If (M^n, g) is a complete Riemannian manifold with $\mathrm{Rc} \geq -(n-1)H$, where $H \geq 0$, and if u is a positive harmonic function on a ball $B(p,r)$, then

$$\sup_{x \in B(p,r/2)} u \leq C \inf_{x \in B(p,r/2)} u$$

where C depends only on n, H and r. In particular, if u is harmonic in $B(p,r) \subset M^n$, then

$$\sup_{x \in B(p,r/2)} |\nabla \log u|(x) \leq C_n \frac{1+\sqrt{H}r}{r}$$

where C_n depends only on n. (See [**443**] for a nice exposition of this.)

Yau's theorem was extended to harmonic functions with sublinear growth (without assuming nonnegativity) by Cheng and Yau [**127**]. They proved the Liouville theorem under the assumption to $|u(x)| = o(d(x,p))$. This result is also sharp in the sense that linear functions on euclidean space are harmonic.

More generally, Yau has conjectured sharp estimates for the dimension of polynomial growth harmonic/holomorphic functions on complete Riemannian/Kähler manifolds with nonnegative Ricci/bisectional curvature. Given a Riemannian manifold (M^n, g), let $\mathcal{H}_d(M^n, g)$ denote the space of harmonic functions u with $|u(x)| \leq C d(x,p)^d$, and given a Kähler manifold (M^n, g), let $\mathcal{O}_d(M^n, g)$ denote the space of holomorphic functions u with $|u(x)| \leq C d(x,p)^d$.

THEOREM 8.9 (Colding-Minicozzi [**166**], Li [**341**]). *If (M^n, g) is complete Riemannian manifold with nonnegative Ricci curvature, then*

$$\dim_{\mathbb{R}} \mathcal{H}_d(M^n, g) < \infty.$$

THEOREM 8.10 (Ni, Chen-Fu-Yin-Zhu [**397**], [**111**]). *If (M^n, g) is complete Riemannian manifold with nonnegative bisectional curvature, then*

$$\dim_{\mathbb{C}} \mathcal{O}_d(M^n, g) \leq \dim_{\mathbb{C}} \mathcal{O}_d(\mathbb{C}^n).$$

If there exists $d \geq 1$ such that $\dim_{\mathbb{C}} \mathcal{O}_d(M^n, g) = \dim_{\mathbb{C}} \mathcal{O}_d(\mathbb{C}^n)$, then (M^n, g) is biholomorphic and isometric to complex euclidean space.

1.1. Another (similar) proof of the Liouville theorem. Suppose $\mathrm{Rc} \geq 0$ and $\Delta f = 0$. Then by (1.61) we have

$$\frac{1}{2}\Delta |\nabla f|^2 \geq |\nabla\nabla f|^2.$$

We compute

$$\frac{1}{2}\Delta\left(\frac{|\nabla f|^2}{f^2}\right) \geq \frac{1}{f^2}\left(|\nabla\nabla f|^2 - \frac{2}{f}\nabla f \cdot \nabla |\nabla f|^2 + \frac{3}{f^2}|\nabla f|^4\right).$$

Hence if we let $u = \log f$, then we obtain $\Delta u = -|\nabla u|^2$ and

$$\frac{1}{2}\Delta |\nabla u|^2 \geq -\nabla u \cdot \nabla |\nabla u|^2 + |\nabla\nabla u|^2 .$$

Given a point $p \in M^n$ and a radius R, let $\eta(x) = \eta(d(x,p))$ where $\eta : [0,\infty) \to [0,1]$ satisfies

$$\eta(r) = \begin{cases} 1 & \text{if } r \leq R \\ 0 & \text{if } r \geq 2R \end{cases},$$

$\eta' \leq 0$, $(\eta')^2 \leq (\eta')^2/\eta \leq CR^{-2}$ and $|\eta''| \leq CR^{-2}$ for some $C < \infty$. We compute that $Q \doteqdot \eta|\nabla u|^2$ satisfies at points where $\eta > 0$

$$\begin{aligned}\frac{1}{2}\Delta Q &\geq \frac{1}{2}\Delta\eta|\nabla u|^2 + \nabla\eta \cdot \nabla|\nabla u|^2 - \eta\nabla u \cdot \nabla|\nabla u|^2 + \eta|\nabla\nabla u|^2 \\ &\geq \left(\frac{1}{2}\eta^{-1}\Delta\eta - \eta^{-2}|\nabla\eta|^2 + \eta^{-1}\nabla\eta\cdot\nabla u\right)Q \\ &\quad + \left(\eta^{-1}\nabla\eta - \nabla u\right)\cdot\nabla Q + \frac{1}{n}\eta^{-1}Q^2\end{aligned}$$

where the second inequality only used $|\nabla\nabla u|^2 \geq \frac{1}{n}(\Delta u)^2 = \frac{1}{n}|\nabla u|^4$. In particular, if $|\nabla u|^2 \not\equiv 0$, then at a maximum point of Q we have $\Delta Q \leq 0$ and $\nabla Q = 0$ so that

$$0 \geq \frac{1}{2}\Delta\eta - \eta^{-1}|\nabla\eta|^2 + \nabla\eta\cdot\nabla u + \frac{Q}{n}.$$

Since

$$|\nabla\eta\cdot\nabla u| \leq \frac{Q}{2n} + \frac{n}{2}\eta^{-1}|\nabla\eta|^2,$$

we have

$$\begin{aligned}\frac{Q}{n} &\leq -\Delta\eta + (n+2)\eta^{-1}|\nabla\eta|^2 \\ &= -\eta'\Delta d - \eta'' + (n+2)\eta^{-1}\left(\eta'\right)^2\end{aligned}$$

By the Laplacian comparison theorem we have $\Delta d \leq \frac{n-1}{d} \leq \frac{n-1}{R}$ where $\eta' \neq 0$. Hence

$$\frac{Q}{n} \leq C(n)R^{-2}.$$

Calabi's trick takes care of the case when the maximum of Q occurs at a cut point of p.

2. Surfaces with positive curvature

When $\chi(M^2) > 0$ the estimate (**??**) helps us prove the infinite time existence of the solution, since it gives a (time dependent) curvature bound; but unlike the $r < 0$ case, it does not directly imply the curvatures converge to a constant exponentially fast. In order to obtain convergence, we need some new estimates. For some mysterious reason, these estimates can be motivated by the notion of a gradient Ricci soliton (M^n, g):

$$R_{ij} + \nabla_i \nabla_j f + \lambda g_{ij} = 0. \tag{8.12}$$

where $f : M^n \to \mathbb{R}$. It is interesting to see what other quantities vanish on gradient Ricci solitons. First, taking the trace, we have

$$R + \Delta f + n\lambda = 0. \tag{8.13}$$

When M^n is closed, we have $\Delta f = r - R$, which is same as the equation (**??**) for the potential function.

Next, let's take the divergence of equation (8.12). By the contracted second Bianchi identity and commuting derivatives, we get in all dimensions:

$$0 = \nabla_j \left(R_{ij} + \nabla_i \nabla_j f + \lambda g_{ij} \right) = \frac{1}{2} \nabla_i R + \nabla_i \Delta f - R_{jijk} \nabla_k f$$

$$= -\frac{1}{2} \nabla_i R + R_{ik} \nabla_k f. \tag{8.14}$$

Substituting (8.12) into the above equation, we obtain

$$0 = \nabla_i \left(R + |\nabla f|^2 + 2\lambda f \right) \tag{8.15}$$

so that

$$R - r + |\nabla f|^2 + 2\lambda f = H + 2\lambda f \tag{8.16}$$

is a constant, where H is the quantity defined in (**??**). This presents us with the motivation for making the computation in (**??**) since the potential function f defined by (**??**) satisfies a linear type heat equation.

Note that taking another divergence of (8.14), we have

$$0 = \Delta R + 2 |\mathrm{Rc}|^2 + 2\lambda R - \langle \nabla R, \nabla f \rangle . \tag{8.17}$$

Another way to see this last equation is to observe that $\frac{\partial}{\partial t} g = \mathcal{L}_{\nabla f} g + 2\lambda g$ implies that for a solution to the Ricci flow

$$\Delta R + 2 |\mathrm{Rc}|^2 = \frac{\partial}{\partial t} R = \mathcal{L}_{\nabla f} R - 2\lambda R = \langle \nabla R, \nabla f \rangle - 2\lambda R.$$

(We obtain the second equality from the diffeomorphism invariance of the scalar curvature and the fact that scales like the inverse of the metric.)

Now let's specialize to the case $n = 2$. Here $R_{ij} = \frac{1}{2} R g_{ij}$ and when $R > 0$, (8.14) implies

$$\nabla_i \log R - \nabla_i f = 0. \tag{8.18}$$

Taking a second divergence and using the definition of f, we obtain $\Delta \log R + R + 2\lambda = 0$. Note that when M^2 is compact, since $2\lambda = -r$ and $\frac{dr}{dt} = r^2$, we have $r \geq -\frac{1}{t}$. Hence on a gradient Ricci soliton with positive curvature we have

$$\Delta \log R + R + \frac{1}{t} \geq 0. \tag{8.19}$$

In the next section we see that the above inequality is true for all complete solutions with bounded positive curvature.

3. Harnack estimate on complete surfaces with positive curvature

3.1. Trace estimate. The consideration of inequality (8.19) puts us in a position to describe the first estimate, which is called the **trace differential Harnack estimate**. Given a solution $\left(M^2, g(t)\right)$ to the *unnormalized* Ricci flow on a closed surface with positive curvature, the **trace Harnack quantity** is defined by

$$Q = \Delta \log R + R + \frac{1}{t} = \frac{\partial}{\partial t} \log R - |\nabla \log R|^2 + \frac{1}{t}. \tag{8.20}$$

The second equality follows from

$$\frac{\partial}{\partial t} R = \Delta R + 2\,|\mathrm{Rc}|^2 = \Delta R + R^2. \tag{8.21}$$

A calculation shows that

$$\frac{\partial}{\partial t} Q \geq \Delta Q + 2\,\langle \nabla \log R, \nabla Q\rangle + Q\left(Q - \frac{2}{t}\right). \tag{8.22}$$

We do not give the details here of the above calculation since a more general calculation is carried out in the proof of Proposition 8.21 (see equations (8.34) and (8.36)). By the maximum principle, we have (see [**258**])

THEOREM 8.11 (Trace differential Harnack on surfaces).

$$\boxed{Q(x,t) \geq 0} \tag{8.23}$$

for all $x \in M^2$ *and* $t > 0$.

REMARK 8.12. *An analogue of the above result was proved in* [**131**] *for solutions on* S^2 *with variable signed curvature (see also* [**275**]*). For a gradient estimate, proved using the method of Aleksandrov reflection, see* [**40**].

An immediate consequence of Theorem 8.11 is:

COROLLARY 8.13 (tR pointwise monotonicity). *For each* $x \in M^2$, *the function* $t \mapsto tR(x,t)$ *is nondecreasing.*

PROOF. Indeed, this follows from $\frac{\partial}{\partial t} \log R + \frac{1}{t} \geq 0$. □

The exact same statement as in the corollary is true in all dimensions under the assumption of bounded, nonnegative curvature operator.

Integrating (8.23) over graphs $(\gamma(t), t)$ of paths $\gamma : [t_1, t_2] \to M^2$ joining points x_1 and x_2, we have

$$\frac{R(x_2,t_2)}{R(x_1,t_1)} \geq \left(\frac{t_2}{t_1}\right)^{-1} \exp\left\{-\frac{1}{4} \inf_{\gamma} \int_{t_1}^{t_2} \left|\frac{d\gamma}{dt}(t)\right|^2_{g(t)} dt\right\}.$$

Let $d(x,y,t)$ denote the **distance** from x to y with respect to the metric $g(t)$. Since $R>0$, we have $g(t)\le g(t_1)$ for all $t\ge t_1$. Hence we obtain the following.

COROLLARY 8.14 (Integral Harnack estimate).

$$\frac{R(x_2,t_2)}{R(x_1,t_1)}\ge\left(\frac{t_2}{t_1}\right)^{-1}\exp\left\{-\frac{d(x_1,x_2,t_1)^2}{4(t_2-t_1)}\right\}.$$

In section **??** below we shall see an application of the Harnack estimate to proving convergence of the Ricci flow when $R(g_0)>0$.

3.2. Curvature changing signs. When R changes sign we can also obtain a trace Harnack estimate. We find it convenient to consider the *normalized* Ricci flow for this estimate. Let $s(t)$ be a solution to the ODE $\frac{ds}{dt}=s^2-rs$ with $s(0)<\min_{x\in M^2}R(x,0)$. This is the ODE corresponding to the PDE (**??**) for the scalar curvature. We find that

$$\frac{\partial}{\partial t}(R-s)=\Delta(R-s)+(R-s)(R-r+s).$$

and the inequality $R-s>0$ is preserved. Let

$$\tilde{Q}\doteqdot\frac{\partial}{\partial t}\log(R-s)-|\nabla\log(R-s)|^2-s=\Delta\log(R-s)+R-r.$$

One can extend the proof of Theorem 8.11 to show the following.

PROPOSITION 8.15 (Trace differential Harnack - curvature changing signs). *There exists $C<\infty$ such that for all $x\in M^2$ and $t\ge0$*

$$\tilde{Q}(x,t)\ge -C.$$

Integrating yields:

COROLLARY 8.16. *There exists $C<\infty$ such that for all $x_1,x_2\in M^2$ and $t_2>t_1\ge0$*

$$R(x_2,t_2)-s(t_2)\ge e^{-\Delta/4-C(t_2-t_1)}(R(x_1,t_1)-s(t_1)),$$

where

$$\Delta=\Delta(x_1,x_2,t_1,t_2)\doteqdot\inf_\gamma\int_{t_1}^{t_2}\left|\frac{d\gamma}{dt}(t)\right|_{g(t)}^2dt,$$

and the infimum is taken over all paths $\gamma:[t_1,t_2]\to M^2$ whose graphs $(\gamma(t),t)$ join (x_1,t_1) to (x_2,t_2).

From the corollary one can show that if one has a uniform diameter bound for the metrics $g(t)$, then there exists $c>0$ such that for any points $x_1,x\in M^2$ and $t\ge0$

$$R(x,t+1)-s(t+1)\ge c(R(x_1,t)-s(t)).$$

Taking $x_1\in M^2$ such that $R(x_1,t)=r$, we have

$$R(x,t+1)\ge c(r-s(t))+s(t+1).$$

Since $\lim_{t\to\infty} s(t) = 0$, taking t sufficiently large, we conclude $R(x, t+1) > 0$ for all $x \in M^2$. Thus, if we can prove a uniform diameter bound, then it will follow that solutions to the Ricci flow on S^2 eventually have positive curvature.

3.3. Matrix estimate. Finally we note that the trace differential Harnack estimate generalizes to a matrix inequality. Under the same assumptions as above, we have:

PROPOSITION 8.17 (Matrix differential Harnack estimate).

$$\boxed{\nabla_i \nabla_j \log R + \frac{1}{2}\left(R + \frac{1}{t}\right) g_{ij} \geq 0.} \tag{8.24}$$

The trace of this estimate is (8.23). In Chapter 8 we shall see the general form of the differential Harnack estimate in all dimensions, which generalizes (8.24).

EXERCISE 8.18. *Prove (8.24). In particular, if we let* $Q_{ij} \doteqdot \nabla_i \nabla_j \log R + \frac{1}{2}\left(R + \frac{1}{t}\right) g_{ij}$ *and* $Q \doteqdot g^{ij} Q_{ij}$, *show that under the Ricci flow on surfaces*

$$\frac{\partial}{\partial t} Q_{ij} = \Delta Q_{ij} + 2\nabla_k Q_{ij} \nabla_k L + 2Q_{ik} Q_{jk} - \left(\frac{2}{t} + 3R\right) Q_{ij} + RQ g_{ij}. \tag{8.25}$$

Then show that the maximum principle for tensors implies $Q_{ij} \geq 0$.

4. Linear trace and interpolated estimates for the Ricci flow on surfaces

There is a version of the **Li-Yau estimate** for the Ricci flow.

PROPOSITION 8.19 (Chow and Hamilton 1997, [**150**]). *If* $(M^2, g(t))$ *is a solution to the Ricci flow on a closed surface with* $R > 0$, *and if* u *is a positive solution to*

$$\frac{\partial u}{\partial t} = \Delta u + Ru,$$

then

$$\boxed{\frac{\partial}{\partial t} \log u - |\nabla \log u|^2 + \frac{1}{t} = \triangle \log u + R + \frac{1}{t} \geq 0.} \tag{8.26}$$

We call this the **linear trace Harnack estimate** (or simply **linear trace estimate**). The following special case was proved earlier:

COROLLARY 8.20 (Hamilton 1988, [**258**]).

$$\boxed{\frac{\partial}{\partial t} \log R - |\nabla \log R|^2 + \frac{1}{t} = \triangle \log R + R + \frac{1}{t} \geq 0.} \tag{8.27}$$

We also have the following **interpolating Harnack estimate** which links the Li-Yau estimate to the linear trace estimate.

PROPOSITION 8.21 (Chow 1998, [**136**]). *Given* $\varepsilon > 0$, *if* $\left(M^2, g(t)\right)$ *is a solution to the* ε***-Ricci flow***

$$\frac{\partial}{\partial t} g_{ij} = -2\varepsilon R_{ij} = -\varepsilon R g_{ij} \tag{8.28}$$

on a closed surface with $R > 0$, *and* u *is a positive solution to*

$$\frac{\partial}{\partial t} u = \triangle u + \varepsilon R u, \tag{8.29}$$

then

$$\frac{\partial}{\partial t} \log u - |\nabla \log u|^2 + \frac{1}{t} = \triangle \log u + \varepsilon R + \frac{1}{t} \geq 0. \tag{8.30}$$

COROLLARY 8.22. *For any* $x_1, x_2 \in M$ *and* $0 < t_1 < t_2$,

$$\frac{u(x_2, t_2)}{u(x_1, t_1)} \geq \left(\frac{t_2}{t_1}\right)^{-\frac{n}{2}} e^{-\ell},$$

where $\ell\left(x_2, t_2; x_1, t_1\right) \doteqdot \inf_\gamma L\left(\gamma\right)$

$$L\left(\gamma\right) \doteqdot \frac{1}{4} \int_{t_1}^{t_2} \left|\frac{d\gamma}{dt}\left(t\right)\right|_{g(t)}^2 dt$$

and where the infimum is taken over all smooth paths $\gamma : [t_1, t_2] \to M^2$ *with* $\gamma\left(t_i\right) = x_i$ *for* $i = 1, 2$.

This corollary, which is analogous to Corollary 8.3, was first obtained by Hamilton in the case where $\varepsilon = 1$ and $u = R$. The proof of the above proposition is not difficult and we now present the details.

PROOF. Since $n = 2$ we have some special evolution equations under Ricci flow. Recall that under (8.28) we have (see (2.3) and Lemma 2.32)

$$\frac{\partial R}{\partial t} = \varepsilon\left(\Delta R + R^2\right) \tag{8.31}$$

$$\frac{\partial}{\partial t}\left(\Delta\right) = \varepsilon R \Delta$$

where the Laplacian Δ is acting on functions. Let

$$P \doteqdot \triangle \log u + \varepsilon\, R$$

where u is a positive solution to (8.29). Using the above formulas, we compute

$$\begin{aligned} \frac{\partial P}{\partial t} &= \frac{\partial}{\partial t}\left(\triangle\right) \log u + \Delta\left(\frac{\partial}{\partial t} \log u\right) + \varepsilon\, \frac{\partial R}{\partial t} \\ &= \Delta\left(\triangle \log u + |\nabla \log u|^2 + \varepsilon\, R\right) + \varepsilon R \Delta \log u + \varepsilon\, \frac{\partial R}{\partial t}. \end{aligned}$$

On the RHS we want to convert as many terms to P as possible, so we rewrite this as:

$$\frac{\partial P}{\partial t} = \Delta P + \Delta\left|\nabla \log u\right|^2 + \varepsilon R P - \varepsilon^2 R^2 + \varepsilon\, \frac{\partial R}{\partial t}. \tag{8.32}$$

We rewrite the second (quadratic) term on the RHS as:

$$\begin{aligned}\Delta |\nabla \log u|^2 &= 2 |\nabla\nabla \log u|^2 + 2\Delta\nabla \log u \cdot \nabla \log u \\ &= 2 |\nabla\nabla \log u|^2 + 2\nabla\Delta \log u \cdot \nabla \log u + 2R_{ij}\nabla_i \log u \nabla_j \log u,\end{aligned}$$

where we used (1.60). Substituting this into (8.32), converting one of the gradient terms to P, and using the identity $R_{ij} = \frac{1}{2}Rg_{ij}$, we have

$$\begin{aligned}\frac{\partial P}{\partial t} &= \Delta P + 2 |\nabla_i\nabla_j \log u|^2 + 2\nabla P \cdot \nabla \log u - 2\varepsilon\nabla R \cdot \nabla \log u \\ &\quad + R |\nabla \log u|^2 + \varepsilon RP - \varepsilon^2 R^2 + \varepsilon \frac{\partial R}{\partial t}\end{aligned} \tag{8.33}$$

Now we define the tensor

$$P_{ij} = \nabla_i\nabla_j \log u + \frac{1}{2}\varepsilon R g_{ij}.$$

The motivation for this is that P_{ij} is both similar to $\nabla_i\nabla_j \log u$, which appears on the RHS of (8.33), and $g^{ij}P_{ij} = P$. (Note also that P_{ij} vanishes on a gradient soliton of the ε-Ricci flow flowing along $\nabla \log u$.) Using this and completing the square for the gradient terms in (8.33), we have when $R > 0$

$$\begin{aligned}\frac{\partial P}{\partial t} &= \Delta P + 2 |P_{ij}|^2 - \varepsilon RP + 2\nabla P \cdot \nabla \log u \\ &\quad + R |\nabla \log u - \varepsilon\nabla \log R|^2 + \varepsilon R \left(\frac{\partial}{\partial t} \log R - \varepsilon |\nabla \log R|^2\right).\end{aligned}$$

Dropping the nonnegative gradient term, using the inequality $2 |P_{ij}|^2 \geq P^2$, adding $1/t$ to P, and using the evolution equation (8.31) for R we obtain

$$\begin{aligned}\frac{\partial}{\partial t}\left(P + \frac{1}{t}\right) &= \Delta\left(P + \frac{1}{t}\right) + 2\nabla\left(P + \frac{1}{t}\right) \cdot \nabla \log u \\ &\quad + \left(P - \frac{1}{t}\right)\left(P + \frac{1}{t}\right) - \varepsilon R\left(P + \frac{1}{t}\right) \\ &\quad + \varepsilon R\left(\varepsilon\left(\Delta \log R + R\right) + \frac{1}{t}\right).\end{aligned} \tag{8.34}$$

When $\varepsilon = 0$, the last term is 0 ; we claim

$$\Delta \log R + R + \frac{1}{\varepsilon t} \geq 0 \tag{8.35}$$

when $\varepsilon > 0$ and $R > 0$. To see this, first consider the case $\varepsilon = 1$ and $u = R$. Since $\frac{\partial R}{\partial t} = \Delta R + R^2$, here we have $\left(M^2, g(t), u = R\right)$ is a solution to (8.28)-(8.29) and $\tilde{P} \doteqdot P = \varepsilon\left(\Delta \log R + R\right)$ with $\varepsilon = 1$. Hence

$$\frac{\partial}{\partial t}\left(\tilde{P} + \frac{1}{t}\right) = \Delta\left(\tilde{P} + \frac{1}{t}\right) + 2\nabla\left(\tilde{P} + \frac{1}{t}\right) \cdot \nabla \log u + \left(\tilde{P} - \frac{1}{t}\right)\left(\tilde{P} + \frac{1}{t}\right). \tag{8.36}$$

By the maximum principle, under the Ricci flow

$$0 \leq \tilde{P} + \frac{1}{t} = \Delta \log R + R + \frac{1}{t}.$$

Hence, for $\varepsilon > 0$, we have that under (8.28), inequality (8.35) holds:

$$\Delta \log R + R + \frac{1}{\varepsilon t} \geq 0.$$

(This supplies the details for the proof of Theorem 8.11 by taking $\varepsilon = 1$.) From this and (8.34) we conclude for all $\varepsilon \geq 0$, $\left(M^2, g(t), u\right)$ satisfies

$$\frac{\partial}{\partial t}\left(P + \frac{1}{t}\right) \geq \Delta\left(P + \frac{1}{t}\right) + 2\nabla\left(P + \frac{1}{t}\right) \cdot \nabla \log u$$
$$+ \left(P - \frac{1}{t}\right)\left(P + \frac{1}{t}\right) - \varepsilon R\left(P + \frac{1}{t}\right).$$

By the maximum principle, since $P + \frac{1}{t} > 0$ for t sufficiently small, we have $P + \frac{1}{t} \geq 0$ for all $t \geq 0$ as long as the solution exists. □

5. Hamilton's matrix estimate

In n-dimensions, we have the following generalization of Hamilton's differential Harnack estimate for surfaces.

THEOREM 8.23 (Hamilton 1993 - Matrix Harnack for the Ricci flow, [**259**]). *If $(M^n, g(t))$ is a solution to the Ricci flow with nonnegative curvature operator, so $R_{ijk\ell}U_{ij}U_{\ell k} \geq 0$ for all 2-forms, and either $(M^n, g(t))$ is compact or complete noncompact with bounded curvature, then for any 1-form $W \in C^\infty\left(\wedge^1 M\right)$ and 2-form $U \in C^\infty\left(\wedge^2 M\right)$ we have*

$$Z(U, W) \doteqdot M_{ij}W_iW_j + 2P_{pij}U_{pi}W_j + R_{pijq}U_{pi}U_{qj} \geq 0. \tag{8.37}$$

Here the 3-tensor P is defined by

$$P_{kij} \doteqdot \nabla_k R_{ij} - \nabla_i R_{kj} \tag{8.38}$$

and the symmetric 2-tensor M is defined by

$$M_{ij} \doteqdot \triangle R_{ij} - \frac{1}{2}\nabla_i\nabla_j R + 2R_{kij\ell}R_{k\ell} - R_{ip}R_{pj} + \frac{1}{2t}R_{ij}. \tag{8.39}$$

We call this **Hamilton's matrix Harnack estimate for the Ricci flow**. Observe the appearance of these tensors in Lemma 9.8 below. In particular,

$$P_{kij} = g_{j\ell}\tilde{R}_{ik0}^{\ell} \tag{8.40}$$

$$M_{ij} = g_{j\ell}\tilde{R}_{i00}^{\ell} + \frac{1}{2t}R_{ij}. \tag{8.41}$$

Note also the identities

$$P_{kij} = \nabla_\ell R_{kij\ell} \tag{8.42}$$

(which is the once contracted second Bianchi identity (1.18)) and

$$M_{ij} = \nabla_k P_{kij} + R_{kij\ell} R_{k\ell} + \frac{1}{2t} R_{ij}. \tag{8.43}$$

Indeed, (8.43) follows from (8.39), commuting a pair of covariant derivatives in the expression for $\nabla_k P_{kij}$, and using the contracted second Bianchi identity:

$$\begin{aligned}\nabla_k P_{kij} &= \nabla_k \left(\nabla_k R_{ij} - \nabla_i R_{kj} \right) \\ &= \Delta R_{ij} - \nabla_i \operatorname{div}\left(\mathrm{Rc}\right)_j + R_{kij\ell} R_{k\ell} - R_{ip} R_{pj}.\end{aligned}$$

The reader may also check that (8.42) and (8.43) are equivalent to the space-time identity (9.31): $\tilde{g}^{ef} \tilde{\nabla}_e \tilde{R}_{abf}^d = \tilde{R}_{ab0}^d$.

The motivation for considering the expression $Z(U, W)$ is as follows. Suppose we have an expanding gradient Ricci soliton (M^n, g) flowing along the vector field ∇f. That is,

$$R_{ij} + \nabla_i \nabla_j f + \frac{1}{2t} g_{ij} = 0. \tag{8.44}$$

Recall that on a compact manifold, we had the coefficient $-\frac{r}{n}$ in front of g_{ij} (see (8.12) and the discussion preceding it). The reason for now considering the coefficient $\frac{1}{2t}$ is that for an Einstein metric with $\lim_{t \to 0} R = -\infty$, we have $R_{ij} + \frac{1}{2t} g_{ij} = 0$. To obtain the Harnack quadratic we differentiate equation (8.44):

$$\nabla_i R_{jk} - \nabla_j R_{ik} = -\nabla_i \nabla_j \nabla_k f + \nabla_j \nabla_i \nabla_k f = R_{ijk\ell} \nabla_\ell f$$

so that

$$P_{ijk} - R_{ijk\ell} \nabla_\ell f = 0. \tag{8.45}$$

Next we take a divergence:

$$\begin{aligned}0 &= \nabla_i P_{ijk} - R_{ijk\ell} \nabla_i \nabla_\ell f - \nabla_i R_{ijk\ell} \nabla_\ell f \\ &= M_{jk} - R_{ijk\ell} R_{i\ell} - \frac{1}{2t} R_{jk} - R_{ijk\ell} \nabla_i \nabla_\ell f + \nabla_i R_{k\ell ji} \nabla_\ell f \\ &= M_{jk} + P_{k\ell j} \nabla_\ell f.\end{aligned}$$

Hence

$$M_{jk} + 2 P_{k\ell j} \nabla_\ell f - R_{k\ell ji} \nabla_i f \nabla_\ell f = 0$$

and for any 1-form W we have

$$M_{jk} W_j W_k - 2 P_{k\ell j} W_k \nabla_\ell f W_j + R_{k\ell ij} W_k \nabla_\ell f \nabla_i f W_j = 0.$$

Now if we take $U_{ij} = \frac{1}{2} \left(\nabla_i f W_j - \nabla_j f W_i \right)$, then

$$M_{jk} W_j W_k + 2 P_{k\ell j} U_{k\ell} W_j + R_{k\ell ij} U_{k\ell} U_{ji} = 0.$$

The LHS is the expression $Z(U, W)$ defined in (8.37). That is, for a expanding gradient Ricci soliton flowing along ∇f, we have

$$Z(\nabla f \wedge W, W) = 0 \tag{8.46}$$

for all 1-forms W.

Tracing the estimate in Theorem 8.23, we obtain:

COROLLARY 8.24 (Trace Harnack for RF). *Under the same hypotheses as the theorem,*

$$\boxed{\frac{\partial R}{\partial t}+\frac{R}{t}+2\nabla_i R V^i+2R_{ij}V^iV^i\geq 0} \tag{8.47}$$

for any vector field V. In particular, taking $V=0$ one obtains

$$\boxed{\frac{\partial}{\partial t}(tR)\geq 0.}$$

Hence, for any $x\in M$ and $0<t_1\leq t_2$, we have

$$R(x,t_2)\geq\frac{t_1}{t_2}R(x,t_1).$$

PROOF. Let $\{\omega^a\}_{a=1}^n$ be an orthonormal coframe. Applying (8.37) to $U=\omega^a\wedge V$ and $W=\omega^a$, we have

$$\begin{aligned}0&\leq\sum_{a=1}^n Z(\omega^a\wedge V,\omega^a)=\sum_{a=1}^n\left(M_{ij}\omega_i^a\omega_j^a-2P_{pij}V_p\omega_i^a\omega_j^a+R_{pijq}V_i\omega_p^aV_j\omega_q^a\right)\\&=g^{ij}M_{ij}-2P_{pij}V_pg^{ij}+R_{pijq}V_iV_jg^{pq}\\&=\frac{1}{2}\triangle R+|R_{k\ell}|^2+\frac{1}{2t}R+\nabla_pRV_p+R_{ij}V_iV_j\end{aligned}$$

and multiplying by 2 yields (8.47). □

EXERCISE 8.25. *Under the hypotheses of Theorem 8.23, show that for any 1-form V*

$$M_{ij}+2P_{pij}V_p+R_{pijq}V_pV_q\geq 0. \tag{8.48}$$

REMARK 8.26. *From (8.46) we see that*

$$H(V)\doteqdot\frac{\partial R}{\partial t}+\frac{R}{t}+2\nabla_iRV^i+2R_{ij}V^iV^i$$

vanishes on an expanding gradient Ricci soliton flowing along $V=\nabla f$.

In singularity analysis, we shall find particularly useful this last fact that the scalar curvature does not decrease too fast.

COROLLARY 8.27 (Ancient solutions with bounded Rm ≥ 0 have R nondecreasing). *If in addition to the above hypotheses, the solution to the Ricci flow exists for all ancient time, i.e., $g(t)$ is defined for $t\in(-\infty,0]$, then*

$$\frac{\partial R}{\partial t}\geq 0. \tag{8.49}$$

PROOF. For any $\alpha < 0$, using the fact that the solution exist on the time interval $[\alpha, 0]$, we have

$$\frac{\partial R}{\partial t} + \frac{R}{t - \alpha} \geq 0.$$

The corollary now follows from taking the limit as $\alpha \to -\infty$. □

For all $n \geq 2$, if $\mathrm{Rc} > 0$, then the minimizing vector field V for the LHS of (8.47) is $V^i = -\frac{1}{2}\left(\mathrm{Rc}^{-1}\right)^{ij} \nabla_j R$, so we get

$$\frac{\partial R}{\partial t} + \frac{R}{t} - \frac{1}{2}\left(\mathrm{Rc}^{-1}\right)^{ij} \nabla_i R \nabla_j R \geq 0.$$

Since $\mathrm{Rc}^{-1} \geq \frac{1}{R} g^{-1}$ (or $\mathrm{Rc} \leq Rg$), we conclude

$$\frac{\partial R}{\partial t} + \frac{R}{t} - \frac{1}{2}\frac{|\nabla R|^2}{R} \geq 0.$$

When $n = 2$, we can get a slightly better estimate by using the identity $R_{ij} = \frac{1}{2} R g_{ij}$. Here we find that

$$\frac{\partial R}{\partial t} + \frac{R}{t} - \frac{|\nabla R|^2}{R} \geq 0,$$

which is equivalent to (8.27).

In fact, we can see that when $n = 2$ (8.37) is equivalent to (8.24). In particular,

$$P_{kij} = \frac{1}{2}\left(\nabla_k R g_{ij} - \nabla_i R g_{kj}\right)$$

and

$$M_{ij} = \frac{1}{2} \triangle R g_{ij} - \frac{1}{2} \nabla_i \nabla_j R + \frac{1}{4} R \left(R + \frac{1}{t}\right) g_{ij}.$$

So (8.48) implies

$$\begin{aligned} \alpha_{ij} &\doteqdot \triangle R g_{ij} - \nabla_i \nabla_j R + \frac{1}{2} R \left(R + \frac{1}{t}\right) g_{ij} \\ &\quad + 2\nabla R \cdot V g_{ij} - \left(\nabla_i R V_j + \nabla_j R V_i\right) + R\left(|V|^2 g_{ij} - V_i V_j\right) \\ &\quad \geq 0. \end{aligned} \tag{8.50}$$

Given a symmetric 2-tensor ω, we define the symmetric 2-tensor $J(\omega)$ by

$$J(\omega)_{ij} \doteqdot \left(\mathrm{trace}_g \omega\right) g_{ij} - \omega_{ij}. \tag{8.51}$$

Diagonalizing ω, we see that if

$$(\omega_{ij}) = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}, \text{ then } \left(J(\omega)_{ij}\right) = \begin{pmatrix} \lambda_2 & 0 \\ 0 & \lambda_1 \end{pmatrix},$$

so that J as rotates the eigenspaces of ω by $90°$. Note

$$\begin{aligned} J\left(\nabla\nabla R\right)_{ij} &= \triangle R g_{ij} - \nabla_i \nabla_j R \\ J\left(\nabla R \otimes V + V \otimes \nabla R\right)_{ij} &= 2\nabla R \cdot V g_{ij} - \left(\nabla_i R V_j + \nabla_j R V_i\right) \\ J\left(V \otimes V\right)_{ij} &= |V|^2 g_{ij} - V_i V_j \\ J\left(\phi g\right)_{ij} &= \phi g_{ij} \end{aligned}$$

for any function ϕ. Hence (8.50) says

$$J\left(\nabla\nabla R + \frac{1}{2}R\left(R + \frac{1}{t}\right)g + \nabla R \otimes V + V \otimes \nabla R + RV \otimes V\right)_{ij} \geq 0.$$

Clearly $\alpha_{ij} \geq 0$ implies $J\left(\alpha\right)_{ij} \geq 0$. Hence

$$\nabla_i \nabla_j R + \frac{1}{2}R\left(R + \frac{1}{t}\right)g_{ij} + \nabla_i R V_j + \nabla_j R V_i + R V_i V_j \geq 0.$$

The minimizing choice is $V = -\frac{1}{R}\nabla R$, so that after dividing by R, we obtain

$$\nabla_i \nabla_j \log R + \frac{1}{2}\left(R + \frac{1}{t}\right)g_{ij} \geq 0,$$

which is the matrix Harnack inequality (8.24).

There is a similar matrix Harnack inequality for solutions to the Kähler-Ricci flow with **nonnegative bisectional curvature** by Cao [**72**] (see Chapter **??**).

6. Sketch of the proof of the matrix Harnack estimate

In this section we briefly sketch the proof of Hamilton's matrix Harnack estimate. For simplicity we shall only consider the case where both M^n is closed and $\mathrm{Rm}\left(g\left(t\right)\right) > 0$. For the general case where $\left(M^n, g\left(t\right)\right)$ is complete with nonnegative curvature operator we ask the reader to consult [**259**] or [**143**]. Related to Uhlenbeck's trick discussed following Lemma 3.11, below we consider tensors as functions on the orthonormal frame bundle; this simplifies the evolution equations for the Harnack quadratic.

Let $\left(M^n, g\right)$ be a Riemannian manifold and OM be its orthonormal frame bundle, which is an $\mathrm{O}\left(n, \mathbb{R}\right)$-principal bundle. Given a covariant p-tensor ω, we define the corresponding function

$$\omega^\dagger : OM \to \mathbb{R}^{n^p}$$

by

$$\omega^\dagger\left(Y\right) \doteqdot \left(\omega_{a_1 \cdots a_p}\left(Y\right)\right)_{a_1,\ldots,a_p=1}^n = \left(\omega\left(Y_{a_1}, \ldots, Y_{a_p}\right)\right)_{a_1,\ldots,a_p=1}^n$$

where $Y = \left\{Y_a\right\}_{a=1}^n$ is an orthonormal frame. This definition provides a 1-1 correspondence between covariant p-tensors ω and $\mathrm{O}\left(n, \mathbb{R}\right)$-equivariant functions $\omega^\dagger : OM \to \mathbb{R}^{n^p}$.

EXERCISE 8.28. *Define* $\omega^\dagger : OM \to \mathbb{R}^{n^{p+q}}$ *for a* (p,q)*-tensor* ω.

Since under the Ricci flow the metric depends on t, it is convenient to perform calculations in local coordinates on the time-dependent orthonormal frame bundle. We shall suppress these calculations and just provide the answers; for details see [**259**] or [**143**]. Given a solution $(M^n, g(t))$, $t \in [0,T)$, of the Ricci flow, we may consider the orthonormal frame bundles $OM(t)$ as contained in $FM \times [0,T)$, where FM is the $\mathrm{GL}(n,\mathbb{R})$-principal frame bundle. The vector field $\partial/\partial t$ is not tangent to the submanifold $O\tilde{M} \doteqdot \cup_{t\in[0,T)} OM(t) \times \{t\}$. To rectify this, we define

$$\nabla_{\partial/\partial t} \doteqdot \frac{\partial}{\partial t} + R_{ab} g^{bc} \Lambda_c^a$$

where the vector fields Λ_c^b on FM are defined via their action on tensors by

$$\left(\Lambda_c^b \omega\right)_{a_1\cdots a_p} \doteqdot \sum_{i=1}^p \delta_{a_i}^b \omega_{a_1\cdots a_{i-1} c a_{i+1}\cdots a_p}.$$

The vector field $\nabla_{\partial/\partial t}$ is tangent to $O\tilde{M}$ and $\nabla_{\partial/\partial t} - \frac{\partial}{\partial t}$ is space-like and perpendicular to $OM(t)$.

EXERCISE 8.29. *Show that*

$$\nabla_{\partial/\partial t} g_{ab} = 0$$
$$\nabla_{\partial/\partial t} R_{ab} = \Delta R_{ab} + 2R_{cabd} R_{cd}.$$

The evolution equations for the components of the Harnack quadratic are given by the following (see Lemmas 4.2, 4.3, and 4.4 of [**259**].). Since these formulas may be derived using the space-time formalism in Chapter 9, we omit the proof.

LEMMA 8.30 (Evolution of components of matrix Harnack quadratic).

$$(\nabla_{\partial/\partial t} - \triangle) R_{abcd} = 2(B_{abcd} - B_{abdc} + B_{acbd} - B_{adbc}), \tag{8.52}$$

$$\begin{aligned}(\nabla_{\partial/\partial t} - \triangle) P_{abc} = &-2R_{de} D_d R_{abec} \\ &+ 2\left(R_{adeb} P_{dec} + R_{adec} P_{dbe} + R_{bdec} P_{ade}\right),\end{aligned} \tag{8.53}$$

and

$$\begin{aligned}(\nabla_{\partial/\partial t} - \triangle) M_{ab} = &\, 2R_{cd}\left(D_c P_{dab} + D_c P_{dba}\right) + 2R_{acdb} M_{cd} \\ &+ 2P_{acd}P_{bcd} - 4P_{acd}P_{bdc} + 2R_{cd}R_{ce}R_{adeb} - \frac{1}{2t} R_{ab},\end{aligned} \tag{8.54}$$

where $B_{abcd} \doteqdot -R_{aebf} R_{cedf}$.

REMARK 8.31. *For (8.52) see (3.18)-(3.19). Caveat: the sign convention we are using for* Rm *is opposite of that in Hamilton's papers.*

When calculating the evolution of the Harnack quadratic

$$Z(U,W) = M_{ab} W_a W_b + 2P_{abc} U_{ab} W_c + R_{abcd} U_{ab} U_{dc} \tag{8.55}$$

it is most convenient to impose the following equations for U and W at a point:

$$(\nabla_{\partial/\partial t} - \Delta) U_{ab} = 0 \tag{8.56}$$

$$\nabla_a U_{bc} = \frac{1}{2}(R_{ab} W_c - R_{ac} W_b) + \frac{1}{4t}(g_{ab} W_c - g_{ac} W_b) \tag{8.57}$$

$$(\nabla_{\partial/\partial t} - \Delta) W_a = \frac{1}{t} W_a \tag{8.58}$$

$$\nabla_a W_b = 0. \tag{8.59}$$

In the application of the maximum principle to show that $Z(U,W) > 0$ we assume there is (x_0, t_0) and U and W (not both 0) at (x_0, t_0) such that $Z(U,W) = 0$. To derive a contradiction we have flexibility of extending U and W to a space-time neighborhood of (x_0, t_0) so that (8.56)-(8.59) hold at (x_0, t_0).

EXERCISE 8.32. *Show that if $(M^n, g(t))$ is an expanding gradient Ricci soliton flowing along ∇f and W is a 1-form satisfying (8.59), then $U = \nabla f \wedge W$ satisfies (8.57). Show also that if W satisfies (8.58), then $U = \nabla f \wedge W$ satisfies (8.56).*

A straightforward calculation using (8.52)-(8.54) yields the following.

LEMMA 8.33 (Evolution of the Harnack quadratic). *If $U(x,t)$ is a 2-form and $W(x,t)$ is a 1-form and (8.56)-(8.59) hold at a point and time, then at that point and time we have*

$$\left(\frac{\partial}{\partial t} - \Delta\right)[Z(U,W)] = 2R_{acdb} M_{cd} W_a W_b - 2P_{acd} P_{bdc} W_a W_b + 8R_{adec} P_{dbe} U_{ab} W_c + 4R_{aefc} R_{befd} U_{ab} U_{cd} + (P_{abc} W_c + R_{abdc} U_{cd})(P_{abe} W_e + R_{abfe} U_{ef}). \tag{8.60}$$

We are now in a position to give the

PROOF OF THEOREM 8.23 WHEN M^n CLOSED AND $\mathrm{Rm} > 0$. It is not difficult to see that for t sufficiently small, we have $Z(U,W) > 0$ for all U and W not both 0. Suppose that $t_0 > 0$ is the first time at which there exists $x_0 \in M^n$ and U and W at x_0 not both 0 such that $Z(U,W) = 0$. Extend U and W in space and time so that (8.56)-(8.59) hold at (x_0, t_0). By the lemma, we have (8.60) at (x_0, t_0). The square term

$$(P_{abc} W_c + R_{abdc} U_{cd})(P_{abe} W_e + R_{abfe} U_{ef})$$

on the last line of (8.60) is nonnegative. We claim the other terms on the RHS of (8.60)

$$J \doteqdot 2R_{acdb} M_{cd} W_a W_b - 2P_{acd} P_{bdc} W_a W_b + 8R_{adec} P_{dbe} U_{ab} W_c + 4R_{aefc} R_{befd} U_{ab} U \tag{8.61}$$

are also nonnegative. Since $Z(U,W)$ is nonnegative at (x_0,t_0), there exist an orthogonal basis of vectors $\left\{Y^N \oplus X^N\right\}_{N=1}^{n(n+1)/2}$ in $\wedge_{x_0}^2 M^n \oplus \wedge_{x_0}^1 M^n$ such that

$$Z(U,W) = \sum_N \left\langle Y^N \oplus X^N, U \oplus W \right\rangle^2 = \sum_N (X_a^N W_a + Y_{ab}^N U_{ab})^2.$$

By (8.55),

$$R_{abdc} = \sum_N Y_{ab}^N Y_{cd}^N \qquad P_{abc} = \sum_N Y_{ab}^N X_c^N \qquad M_{ab} = \sum_N X_a^N X_b^N. \tag{8.62}$$

Substituting these equations into (8.61) yields

$$\begin{aligned} J &= \sum_N Y_{ac}^N Y_{bd}^N \sum_M X_c^M X_d^M W_a W_b - 2\sum_N Y_{ac}^N X_d^N \sum_M Y_{bd}^M X_c^M W_a W_b \\ &\quad + 8\sum_N Y_{ad}^N Y_{ce}^N \sum_M Y_{db}^M X_e^M U_{ab} W_c + 4\sum_N Y_{ae}^N Y_{cf}^N \sum_M Y_{be}^M Y_{df}^M U_{ab} U_{cd} \\ &= \sum_{M,N} (Y_{ac}^M X_c^N W_a - Y_{ac}^N X_c^M W_a - 2Y_{ac}^M Y_{bc}^N U_{ab})^2 \geq 0. \end{aligned}$$

We conclude that if U and W satisfy (8.56)-(8.59) and $Z(U,W) \geq 0$ at point and time, then $\left(\frac{\partial}{\partial t} - \Delta\right)[Z(U,W)] \geq 0$ at that point and time. With a little more work, one can show $Z(U,W) \geq 0$ for all U and W at all points and times. □

7. Linear trace Harnack estimate in all dimensions

Recall that the Lichnerowicz Laplacian acting on symmetric 2-tensors is given by

$$\Delta_L v_{ij} = \Delta v_{ij} + 2R_{kij\ell} v_{k\ell} - R_{ik} v_{jk} - R_{jk} v_{ik}.$$

We have the following **linear trace Harnack estimate**.

THEOREM 8.34 (Chow and Hamilton 1997). *If $(M^n, g(t), v(t))$, $t \in [0,T)$ is a solution to the linearized Ricci flow system:*

$$\frac{\partial}{\partial t} g_{ij} = -2R_{ij} \tag{8.63}$$

$$\frac{\partial}{\partial t} v_{ij} = \Delta_L v_{ij} \tag{8.64}$$

where $(M^n, g(t))$ is complete with bounded, nonnegative curvature operator and $v(t)$ is also nonnegative and bounded, then $v(t) \geq 0$ for $t \in [0,t)$ and for any vector X we have

$$\boxed{Z(X) \doteqdot \nabla_i \nabla_j v_{ij} + R_{ij} v_{ij} + 2(\nabla_j v_{ij}) X_i + v_{ij} X_i X_j + \frac{V}{2t} \geq 0} \tag{8.65}$$

where $V = g^{ij} v_{ij}$.

REMARK 8.35. *The linear trace Harnack estimate has a Kähler analogue which we discuss in Chapter* **??**.

When $n = 2$, we may set $v_{ij} = ug_{ij}$ and we then find that u satisfies the equation

$$\frac{\partial u}{\partial t} = \Delta u + Ru.$$

In this case, inequality (8.65) reduces to

$$\Delta u + Ru + 2\nabla u \cdot X + u\,|X|^2 + \frac{u}{t} \geq 0$$

for all vector fields X. If $u > 0$, we then have

$$u\left(\Delta \log u + R + \frac{1}{t}\right) + u\,|\nabla \log u + X|^2 \geq 0.$$

Hence, by taking $X = -\nabla \log u$, we get the equivalent inequality:

$$\Delta \log u + R + \frac{1}{t} \geq 0.$$

This is (8.26).

Proof of Theorem 8.34. (*Sketch.*) We just give the proof for the case where $v_{ij} > 0$ and $V = g^{ij}v_{ij} \geq \varepsilon$ for some $\varepsilon > 0$. In this case the vector field Y minimizing Z in (8.65) satisfies $\operatorname{div}(v)_i + v_{ij}Y_j = 0$, or equivalently $Y^j = -\left(v^{-1}\right)^{ji}\operatorname{div}(v)_i$. One can calculate that (see [**150**] or [**147**] for details)

(8.66)

$$\begin{aligned}\frac{\partial}{\partial t}Z(Y) &= \triangle Z(Y) + 2v_{ij}\left(M_{ij} + 2P_{pij}Y_p + R_{pijq}Y_pY_q\right)\\ &\quad + 2v_{ij}\left(\nabla_k Y_i - R_{ki} - \frac{1}{2t}g_{ki}\right)\left(\nabla_k Y_j - R_{kj} - \frac{1}{2t}g_{ki}\right) - \frac{2}{t}Z(Y),\end{aligned}$$

where M_{ij} and P_{pij} are defined in (8.39) and (8.38), respectively. Recall also that

$$Z(U,W) \doteqdot M_{ij}W_iW_j + 2P_{pij}U_{pi}W_j + R_{pijq}U_{pi}U_{qj}$$

is the the matrix Harnack quadratic. Since $v > 0$, at each point there exists an orthonormal coframe $\{\omega^a\}_{a=1}^n$ and $\lambda_a > 0$ such that $v = \sum_{a=1}^n \lambda_a \omega^a \otimes \omega^a$. Then the second term on the RHS of (8.66) is

$$v_{ij}\left(M_{ij} + 2P_{pij}Y_p + R_{pijq}Y_pY_q\right) = \sum_{a=1}^{n} \lambda_a Z\left(Y \wedge \omega^a, \omega^a\right).$$

By the matrix Harnack estimate, this term is nonnegative. (Equivalently, we could have used (8.48).) Thus

$$\frac{\partial}{\partial t}Z(Y) \geq \triangle Z(Y) - \frac{2}{t}Z(Y). \tag{8.67}$$

Since $V \geq \varepsilon > 0$, we have $Z(Y) > 0$ for t small enough. Applying the maximum principle to (8.67), we conclude that $Z(Y) \geq 0$. $\square$

A fundamental application of the linear trace Harnack estimate is the following classification of eternal solutions. Originally, this result was proved by Hamilton in [**260**] using the matrix Harnack estimate. The proof below we give is due to Ni [**398**] (an analogous result also holds for immortal solutions).

THEOREM 8.36 (Ancient solution with $\mathrm{Rm} \geq 0$ and attaining $\sup R$ are steady gradient solitons). *If $(M^n, g(t))$, $t \in (-\infty, \omega)$, is a complete solution to the Ricci flow with nonnegative curvature operator, positive Ricci curvature, and such that $\sup_{M\times(-\infty,\omega)} R$ is attained at some point in space and time, then $(M^n, g(t))$ is a steady gradient Ricci soliton.*

REMARK 8.37. *This is the generalization to all dimensions of Theorem 7.16.*

PROOF. Let

$$\bar{Z}[v] = \bar{Z}[v](Y) \doteqdot \nabla_i\nabla_j v_{ij} + R_{ij}v_{ij} + 2(\nabla_j v_{ij})Y_i + v_{ij}Y_iY_j, \tag{8.68}$$

where $Y^j = -\left(v^{-1}\right)^{ji}\operatorname{div}(v)_i$. Then similar to (8.66) we have

$$\begin{aligned}\frac{\partial}{\partial t}\bar{Z}[v] &= \triangle\bar{Z}[v] + 2v_{ij}\left(\bar{M}_{ij} + 2P_{pij}Y_p + R_{pijq}Y_pY_q\right) \\ &\quad + 2v_{ij}\left(\nabla_k Y_i - R_{ki}\right)\left(\nabla_k Y_j - R_{kj}\right),\end{aligned} \tag{8.69}$$

where

$$\bar{M}_{ij} \doteqdot \triangle R_{ij} - \frac{1}{2}\nabla_i\nabla_j R + 2R_{kij\ell}R_{k\ell} - R_{ip}R_{pj} = M_{ij} - \frac{1}{2t}R_{ij}. \tag{8.70}$$

Since $(M^n, g(t))$ is an ancient solution, the matrix Harnack estimate says

$$\bar{M}_{ij} + 2P_{pij}Y_p + R_{pijq}Y_pY_q \geq 0.$$

If $v > 0$, then $v = \sum_{a=1}^n \lambda_a W^a \otimes W^a$ where $\{W^a\}$ is orthonormal and $\lambda_a > 0$. Suppose there exists a point and time (x_0, t_0) such that $\bar{Z}[v](x_0, t_0) = 0$. Then by the strong maximum principle, $\bar{Z}[v] \equiv 0$ and

$$0 \equiv v_{ij}\left(\nabla_k Y_i - R_{ki}\right)\left(\nabla_k Y_j - R_{kj}\right) = \sum_{a,b=1}^n \lambda_a\left[(\nabla Y - \mathrm{Rc})\left(W^b, W^a\right)\right]^2.$$

Since $\lambda_a > 0$, this implies

$$\nabla_i Y_j - R_{ij} = 0$$

and we conclude that $(M^n, g(t))$ is a steady Ricci soliton. Now we need to find v such that the condition $\bar{Z}[v](x_0, t_0) = 0$ holds for some (x_0, t_0). To do this, simply take $v = \mathrm{Rc} > 0$. Since we have assumed $\sup_{M\times(-\infty,\infty)} R$ is attained at some point in space and time, there exists (x_0, t_0) such that $\frac{\partial R}{\partial t} = 0$ and $\nabla R = 0$ at (x_0, t_0). This implies $\bar{Z}(x_0, t_0) = 0$, and we can apply the above argument. □

EXERCISE 8.38. *Establish the following formula which generalizes (8.69). If $(M^n, g(t))$ is a solution to Ricci flow and $v_{ij}(t)$ is any symmetric 2-tensor and $X(t)$ any vector field, then*

$$\begin{aligned}
&\left(\frac{\partial}{\partial t} - \Delta\right)\left(\nabla^i\nabla^j v_{ij} + R_{ij}v_{ij} - 2X^i\nabla^j v_{ij} + v_{ij}X^iX^j\right) \\
&= \left(\nabla^i\nabla^j + R_{ij} - 2X^i\nabla^j + X^iX^j\right)\left(\left(\frac{\partial}{\partial t} - \Delta_L\right)v_{ij}\right) \\
&+ 2\begin{pmatrix} \Delta R_{ij} - \frac{1}{2}\nabla_i\nabla_j R + 2R_{kij\ell}R_{k\ell} - R_{ik}R_{jk} \\ +2\left(\nabla_i R_{j\ell} - \nabla_\ell R_{ji}\right)X^\ell + R_{kij\ell}X^kX^\ell \end{pmatrix} v_{ij} \\
&+ 2\left(R_{ik} + \nabla_k X_i\right)\left(R_{jk} + \nabla_k X_j\right)v_{ij} \\
&+ 2\left(-\left(\frac{\partial}{\partial t} - \Delta\right)X^i + X^\ell R_{i\ell} + 2\left(R_{ik} + \nabla_k X_i\right)\cdot\nabla_k\right)\left(\nabla^j v_{ij} - v_{ij}X^j\right).
\end{aligned} \tag{8.71}$$

Note that when $v_{ij}(t)$ is a solution to the Lichnerowicz Laplacian heat equation, the first term on the RHS *is zero. Also, the vector field X minimizing the linear trace Harnack quadratic satisfies $\nabla^j v_{ij} - v_{ij}X^j = 0$ in which case the last line is zero.*

REMARK 8.39. *It is interesting to note that if $\tilde{\nabla}^i \doteqdot \nabla^i - X^i$ and $\tilde{R}_{ij} \doteqdot R_{ij} + \nabla_i X_j$, then*

$$\nabla^i\nabla^j + R_{ij} - 2X^j\nabla^i + X^iX^j = \tilde{\nabla}^i\tilde{\nabla}^j + \tilde{R}_{ij}, \tag{8.72}$$

so that the linear trace Harnack calculation can be interpreted as

$$\begin{aligned}
&\left(\frac{\partial}{\partial t} - \Delta\right)\left(\left(\tilde{\nabla}^i\tilde{\nabla}^j + \tilde{R}_{ij}\right)v_{ij}\right) \\
&= \left(\tilde{\nabla}^i\tilde{\nabla}^j + \tilde{R}_{ij}\right)\left(\left(\frac{\partial}{\partial t} - \Delta_L\right)v_{ij}\right) + 2\left(\bar{M}_{ij} + 2P_{i\ell j}X^\ell + R_{kij\ell}X^kX^\ell\right)v_{ij} \\
&+ 2\tilde{R}_{ki}\tilde{R}_{kj}v_{ij} + 2\left(-\left(\frac{\partial}{\partial t} - \Delta - \mathrm{Rc}\right)X^i + 2\tilde{R}_{ki}\cdot\nabla_k\right)\left(\tilde{\nabla}^j v_{ij}\right).
\end{aligned}$$

HINT. Formula (8.71) follows from combining the following three formulas:

$$\begin{aligned}
\left(\frac{\partial}{\partial t} - \Delta\right)\left(\nabla^i\nabla^j v_{ij} + R_{ij}v_{ij}\right) &= \left(\nabla^i\nabla^j + R_{ij}\right)\left(\left(\frac{\partial}{\partial t} - \Delta_L\right)v_{ij}\right) \\
&+ 2\left(\Delta R_{ij} - \frac{1}{2}\nabla_i\nabla_j R + 2R_{kij\ell}R_{k\ell}\right)v_{ij} \\
&+ 4R_{ik}\nabla_k\nabla_j v_{ij}
\end{aligned}$$

and

$$\left(\frac{\partial}{\partial t}-\Delta\right)\left(X^i\nabla^j v_{ij}\right)$$
$$=X^i\left(\nabla^j\left(\frac{\partial}{\partial t}-\Delta_L\right)v_{ij}+2\left(\nabla_i R_{j\ell}-\nabla_\ell R_{ji}\right)v_{\ell j}+2R_{k\ell}\nabla_k v_{i\ell}-R_{i\ell}\nabla_j v_{j\ell}\right)$$
$$+\left(\left(\frac{\partial}{\partial t}-\Delta\right)X^i\right)\nabla^j v_{ij}-2\nabla_k X^i\nabla_k\nabla^j v_{ij}$$

and

$$\frac{\partial}{\partial t}\left(v_{ij}X^iX^j\right)=\Delta\left(v_{ij}X^iX^j\right)+\left(\left(\frac{\partial}{\partial t}-\Delta\right)v_{ij}\right)X^iX^j$$
$$+2v_{ij}\left(\left(\frac{\partial}{\partial t}-\Delta\right)X^i\right)X^j-2v_{ij}\nabla_kX^i\nabla_kX^j-4\nabla_kv_{ij}\nabla_kX^iX^j.$$

8. A pinching estimate for solutions of the linearized Ricci flow equation

In dimension 3, solutions to the linearized Ricci flow equation satisfy the following a priori estimate. Recall that if $(M^n, g(t))$ is a solution to the Ricci flow on a closed manifold with $R_{\min}(g(0)) > -\rho$, then $R_{\min}(g(t)) > -\rho$ for all $t \geq 0$.

THEOREM 8.40 (Anderson and Chow - 3-d estimate for linearized RF, [**8**]). *If $\left(M^3, g(t), v(t)\right)$, $t \in [0,T)$, $T < \infty$, is a solution to the linearized Ricci flow (8.63)-(8.64) on a closed 3-manifold such that $R_{\min}(g(0)) > -\rho$, then there exists a constant $C < \infty$ depending only on $g(0)$, $v(0)$, ρ and T such that*

$$|v| \leq C(R+\rho) \tag{8.73}$$

on $M \times [0,T)$. If $\rho = 0$, then we may choose C independent of T.

Since we may take $v = \mathrm{Rc}$, one of the estimates in [**255**] is a special case; the 'Ricci pinching preserved' type estimate (3.29).

COROLLARY 8.41 (Hamilton 1982). *If $\left(M^3, g(t)\right)$, $t \in [0,T)$ is a solution to the Ricci flow on a closed 3-manifold with positive scalar curvature, then there exists a constant $C < \infty$ such that $\frac{|\mathrm{Rc}|}{R} \leq C$ on $M^3 \times [0,T)$.*

PROOF OF THEOREM. (*Sketch.*) Let $V = g^{ij}v_{ij}$. One computes that

$$\frac{\partial}{\partial t}\left(\frac{|v|^2}{(R+\rho)^2}\right)=\Delta\left(\frac{|v|^2}{(R+\rho)^2}\right)+\frac{2}{R+\rho}\nabla R\cdot\nabla\left(\frac{|v|^2}{(R+\rho)^2}\right)$$
$$-\frac{2}{(R+\rho)^4}\left|(R+\rho)\nabla_i v_{jk}-\nabla_i R v_{jk}\right|^2+4P,$$

where

(8.74)

$$P \doteqdot \frac{1}{(R+\rho)^3}\left[2R\operatorname{Rc}\cdot vV - 2R\operatorname{Rc}\cdot v^2 + \frac{1}{2}R^2\left(|v|^2 - V^2\right) - |v|^2|\operatorname{Rc}|^2\right] + \frac{\rho}{(R+\rho)^3}\left[2\operatorname{Rc}\cdot vV - 2\operatorname{Rc}\cdot v^2 + \frac{1}{2}R\left(|v|^2 - V^2\right)\right].$$

(Here $\left(v^2\right)_{ij} = v_{ik}v_{jk}$.) One can also show that

$$|v|^2|\operatorname{Rc}|^2 - 2RV\operatorname{Rc}\cdot v + 2R\operatorname{Rc}\cdot v^2 + \frac{1}{2}R^2\left(V^2 - |v|^2\right) \geq 0.$$

Thus we have

$$\frac{\partial}{\partial t}\left(\frac{|v|^2}{(R+\rho)^2}\right) \leq \Delta\left(\frac{|v|^2}{(R+\rho)^2}\right) + \frac{2}{R+\rho}\nabla R\cdot\nabla\left(\frac{|v|^2}{(R+\rho)^2}\right) + 4\frac{\rho}{(R+\rho)^3}\left[2\operatorname{Rc}\cdot vV - 2\operatorname{Rc}\cdot v^2 + \frac{1}{2}R\left(|v|^2 - V^2\right)\right]. \tag{8.75}$$

By the Hamilton-Ivey estimate, we have $|\operatorname{Rc}| \leq C(R+\rho)$. This implies

$$\frac{\rho}{(R+\rho)^3}\left[2\operatorname{Rc}\cdot vV - 2\operatorname{Rc}\cdot v^2 + \frac{1}{2}R\left(|v|^2 - V^2\right)\right] \leq C\rho\frac{|v|^2}{(R+\rho)^2},$$

so that (8.75) becomes

(8.76)

$$\frac{\partial}{\partial t}\left(\frac{|v|^2}{(R+\rho)^2}\right) \leq \Delta\left(\frac{|v|^2}{(R+\rho)^2}\right) + \frac{2\nabla R}{R+\rho}\cdot\nabla\left(\frac{|v|^2}{(R+\rho)^2}\right) + 4C\rho\frac{|v|^2}{(R+\rho)^2}.$$

Applying the maximum principle to this equation implies

$$\frac{|v|^2}{(R+\rho)^2}(t) \leq C_0\exp(4C\rho T),$$

for $t \in [0,T)$, where $C_0 = \max_{t=0}|v|^2/(R+\rho)^2$. □

9. Notes and commentary

§1. For an exposition of differential Harnack estimates for geometric evolution equations, see Hamilton [**273**] and Chow-Guenther [**147**]. For some differential Harnack estimates for the heat equation not discussed in this chapter, see Hamilton [**261**], Yau [**530**], [**531**], and Bakry-Qian [**36**]. Hamilton's matrix Harnack estimate for the heat equation is:

THEOREM 8.42 (Matrix Harnack for heat equation - Hamilton 1993, [**261**]). *If u is a positive solution to the heat equation on a closed Riemannian* *manifold with parallel Ricci tensor and nonnegative sectional curvature, then*

$$\boxed{\nabla_i \nabla_j \log u + \frac{1}{2t} g_{ij} \geq 0.} \tag{8.77}$$

Now if $\gamma(s)$ is a geodesic parametrized by arc length, then this implies

$$\frac{d^2}{ds^2}\left(\log u(\gamma(s)) + \frac{s^2}{4t}\right) \geq 0.$$

An interesting property that smooth solutions of the heat equation on euclidean space satisfy is the Mean Value Property (see p. 51-54 of [**201**]). Given $x_0 \in \mathbb{R}^n$, $t_0 \in \mathbb{R}$ and $r > 0$, define the **heat ball**:

$$E(x_0, t_0; r) \doteqdot \left\{(x,t) \in \mathbb{R}^{n+1} : t \leq t_0,\ (4\pi(t_0 - t))^{n/2} e^{-|x_0 - x|^2/4(t_0-t)} \geq r^{-n}\right\}.$$

THEOREM 8.43 (Mean value property for euclidean heat equation). *Let $U \subset \mathbb{R}^n$ be an open set. If u is a smooth solution to the heat equation in $U \times (0,T]$ and if $E(x_0,t_0;r) \subset U \times (0,T]$, then*

$$u(x_0,t_0) = \frac{1}{4r^n} \iint_{E(x_0,t_0;r)} u(x,t) \frac{|x_0 - x|^2}{(t_0 - t)^2} dxdt.$$

§2. The Ricci soliton equation appears in Friedan [**213**], equation (2.2.1) on p. 395, where the non-Einstein solutions are called **quasi-Einstein metrics**.

§3. The analogue of the trace differential Harnack estimate for the **mean curvature flow** (3.51) $\frac{\partial X}{\partial t} = -H\nu$ is as follows. If we have a weakly convex solution (i.e., the second fundamental form h_{ij} is nonnegative) then (see [**269**])

$$\boxed{\frac{\partial H}{\partial t} + \frac{H}{2t} + 2\langle \nabla H, V\rangle + h_{ij} V^i V^j \geq 0} \tag{8.78}$$

for any vector field V. If $h_{ij} > 0$, then we conclude

$$\frac{\partial H}{\partial t} + \frac{H}{2t} - \left(h^{-1}\right)^{ij} \nabla_i H \nabla_j H \geq 0.$$

Integrating this differential Harnack we obtain the following classical type Harnack inequality. Given $0 < t_1 < t_2$ and $x_i \in X_{t_i}(M)$, $i = 1, 2$, let

$$\Delta = \Delta(x_1, t_1, x_2, t_2) \doteqdot \inf_\gamma \int_{t_1}^{t_2} |\dot{\gamma}_M(t)|^2 dt,$$

where the infimum is taken over all $\gamma : [t_1, t_2] \to \mathbb{R}^n$ with $\gamma(t) \in X_t(M)$, $\gamma(t_i) = x_i$, and

$$\dot{\gamma}_M \doteqdot \frac{d\gamma}{dt} - \left\langle \frac{d\gamma}{dt}, \nu \right\rangle \nu.$$

We have

$$\boxed{H(x_2,t_2) \geq \left(\frac{t_2}{t_2}\right)^{-1/2} e^{-\Delta/4} H(x_1,t_1).}$$

A solution to the mean curvature flow on a finite time interval $[0,T)$ forms a **Type II singularity** if $\sup_{M^{n-1}\times[0,T)} (T-t)|h_{ij}|^2 = \infty$. In this case, assuming an injectivity radius estimate, there exists a complete limit solution $X_\infty(t) : M_\infty^{n-1} \to \mathbb{R}^n$, $t \in (-\infty,\infty)$, which is weakly convex and where the mean curvature attains its maximum at some point and time. Now (8.78) implies

$$\frac{\partial H}{\partial t} + 2\langle \nabla H, V\rangle + h_{ij}V^i V^j \geq 0$$

for all V, since the solution is defined on an ancient time interval. We then can conclude that $X_\infty(t)$ is a **translating soliton**.

There are similar Harnack estimates for the **Gauss curvature flow** (**GCF**). If $X : M^{n-1} \to \mathbb{R}^n$ satisfies $\frac{\partial X}{\partial t} = -K\nu$, where K is the Gauss curvature, then [**133**]:

$$\frac{\partial K}{\partial t} + \frac{(n-1)K}{nt} - \left(h^{-1}\right)^{ij} \nabla_i K \nabla_j K \geq 0.$$

Similar to the mean curvature flow, one shows that

$$\boxed{K(x_2,t_2) \geq \left(\frac{t_2}{t_2}\right)^{-n-1/n} e^{-\Theta/4} K(x_1,t_1)}$$

where

$$\Theta = \Theta(x_1,t_1,x_2,t_2) \doteqdot \inf_\gamma \int_{t_1}^{t_2} Kh^{-1}(\dot\gamma_M,\dot\gamma_M)\,dt.$$

It is particularly useful to choose γ to be a straight line segment in $\mathbb{R}^n$ in which case we obtain the following result of Hamilton [**263**]. If $\gamma : [0,\tau] \to \mathbb{R}^n$ is a straight line segment transversal to $X_t(M)$ for $t \in [0,\tau]$, then

$$t \mapsto t^{(n-1)/n} K(\gamma(t),t)\sec\theta(t)$$

is nondecreasing, where $\theta(t) \in [0,\pi/2)$ is the angle between $\gamma(t)$ and the normal ν.

For much more general curvature flows of hypersurfaces see Andrews [**13**].

§6. One way of computing the evolution of the Laplacian of the Ricci tensor ΔR_{ij} is to use the following.

EXERCISE 8.44. *Show that if a_{ij} is a 2-tensor, then*

$$\begin{aligned}\frac{\partial}{\partial t}(\Delta a_{ij}) = \Delta\left(\frac{\partial}{\partial t}a_{ij}\right) &+ 2R_{kl}\nabla_k\nabla_l a_{ij} \\ &+ 2\left(\nabla_k R_{im} + \nabla_i R_{km} - \nabla_m R_{ki}\right)\nabla_k a_{mj} \\ &+ 2\left(\nabla_k R_{jm} + \nabla_j R_{km} - \nabla_m R_{kj}\right)\nabla_k a_{im} \\ &+ a_{mj}\Delta R_i^m + a_{im}\Delta R_j^m.\end{aligned}$$

SOLUTION. We compute

$$\begin{aligned}\frac{\partial}{\partial t}(\Delta a_{ij}) &= \Delta\left(\frac{\partial}{\partial t}a_{ij}\right) + 2R_{kl}\nabla_k\nabla_l a_{ij} \\ &\quad - \left(\frac{\partial}{\partial t}\Gamma_{kk}^m\right)\nabla_m a_{ij} - \left(\frac{\partial}{\partial t}\Gamma_{ki}^m\right)\nabla_k a_{mj} - \left(\frac{\partial}{\partial t}\Gamma_{kj}^m\right)\nabla_k a_{im} \\ &\quad - \nabla_k\left(\left(\frac{\partial}{\partial t}\Gamma_{ki}^m\right)a_{mj} + \left(\frac{\partial}{\partial t}\Gamma_{kj}^m\right)a_{im}\right) \\ &= \Delta\left(\frac{\partial}{\partial t}a_{ij}\right) + 2R_{kl}\nabla_k\nabla_l a_{ij} \\ &\quad - 2\left(\frac{\partial}{\partial t}\Gamma_{ki}^m\right)\nabla_k a_{mj} - 2\left(\frac{\partial}{\partial t}\Gamma_{kj}^m\right)\nabla_k a_{im} \\ &\quad - a_{mj}\nabla_k\left(\frac{\partial}{\partial t}\Gamma_{ki}^m\right) - a_{im}\nabla_k\left(\frac{\partial}{\partial t}\Gamma_{kj}^m\right)\end{aligned}$$

since $\sum_k \frac{\partial}{\partial t}\Gamma_{kk}^m = 0$. Now

$$-g^{jk}\nabla_j\left(\frac{\partial}{\partial t}\Gamma_{ik}^\ell\right) = \nabla_k\left(\nabla_i R_{kl} + \nabla_k R_{il} - \nabla_l R_{ik}\right) = \Delta R_{il} \tag{8.79}$$

because $\nabla_k\left(\nabla_i R_{kl} - \nabla_l R_{ik}\right) = 0$.

Note that formula (8.79) provides a quick way to derive the evolution of the Ricci tensor (2.37) since from tracing (3.8) we have

$$g^{jk}\frac{\partial}{\partial t}R_{ijk}^\ell = -g^{jk}\nabla_j\left(\frac{\partial}{\partial t}\Gamma_{ik}^\ell\right) = \Delta R_{il}$$

so that

$$\begin{aligned}\frac{\partial}{\partial t}R_i^\ell &= g^{jk}\left(\frac{\partial}{\partial t}R_{ijk}^\ell\right) + \left(\frac{\partial}{\partial t}g^{jk}\right)R_{ijk}^\ell \\ &= \Delta R_{il} + 2R_{jk}R_{ijk}^\ell.\end{aligned}$$

§8. The pinching estimate (8.73) was partially motivated by the work of Gursky [**251**].

CHAPTER 9

Space-time geometry

In this chapter we shall discuss the space-time geometry and its use to both give a geometric interpretation of the Harnack type estimates and to motivate some of Perelman's constructions. In §1 we begin with a space-time connection of Sun-Chin Chu and one of the authors, which when paired with the compatible degenerate space-time metric, satisfies a generalized Ricci flow (see Proposition 9.4). We then calculate the Riemann and Ricci curvature tensors of the space-time connection and observe identities which are essentially the Ricci soliton equation (Lemma 9.11 and equation (9.29).) In §2 we observe that Riemann and Ricci curvature tensors of the space-time connection are Hamilton's matrix and trace Harnack quadratics, respectively (with or without the time terms $\frac{1}{2t}R_{ij}$ and $\frac{R}{t}$, depending on the connection we consider.) We also show that the space-time second Bianchi identity yields a quick way to directly derive the evolution equations for the matrix Harnack quadratic. In §3 we introduce potentially infinite metrics on the product of space-time with an N-dimensional Einstein solution of Ricci flow. The time components of these metrics are $\left(R + \frac{bN}{2t}\right) dt^2$. We calculate the Levi-Civita connection and curvatures of these metrics (Lemmas 9.19, 9.23 and Proposition 9.27) and find that in a certain sense it is best to take $b = -1$ which are Perelman's metrics. In this case the metrics are *potentially Ricci flat* (Theorem 9.31). Next we observe that if we let $N \to \infty$, then the Levi-Civita connections approach the space-time connection defined in §1. More generally, when $b \neq 0$, we show that the space-time metrics potentially satisfy the gradient Ricci soliton equation; only when $b = -1$ is this equation the potentially Ricci flat equation. In §4 we derive the definition of the $\mathcal{L}$-function by renormalizing the length functional of the potentially infinite metrics. Finally in §5 we consider potentially infinite metrics adapted to 1-parameter families of solutions of the Ricci-DeTurck flow. This leads to a geometric space-time interpretation of the linear trace Harnack estimate and is related to Perelman's idea of fixing a measure.

1. A space-time solution to the Ricci flow for degenerate metrics

We begin by considering a generalization of the Ricci flow equation for a metric-connection pair. Recall:

LEMMA 9.1. *Under the Ricci flow the metric inverse and Levi-Civita connection evolve by*

$$\frac{\partial}{\partial t} g^{ij} = 2 g^{ik} g^{j\ell} R_{k\ell} \tag{9.1}$$

$$\frac{\partial}{\partial t} \Gamma_{ij}^{k} = -g^{k\ell} \left(\nabla_i R_{j\ell} + \nabla_j R_{i\ell} - \nabla_\ell R_{ij} \right). \tag{9.2}$$

PROOF. Equation (9.1) follows directly from the Ricci flow equation and the formula for differentiating the inverse of a matrix:

$$\frac{d}{dt} \left(A^{-1} \right)_{ij} = - \left(A^{-1} \right)_{ik} \left(\frac{d}{dt} A_{k\ell} \right) \left(A^{-1} \right)_{\ell j}.$$

Equation (9.2) is just (2.18). □

The **metric** $g^{-1} = g^{ij} \frac{\partial}{\partial x^i} \otimes \frac{\partial}{\partial x^j}$ **on the cotangent bundle** TM^* is positive definite and as such has a unique torsion free compatible connection, the Levi-Civita connection ∇, whose components are the Christoffel symbols.

Given a manifold $\tilde{M}^{n+1}$ and a 1-parameter family of **degenerate** (i.e., positive semi-definite) metrics $\tilde{g}(t)$ on $T\tilde{M}^*$, we generalize the Ricci flow equation as follows.[1]

DEFINITION 9.2 (Hamilton). *A triple* $\left(\tilde{M}^{n+1}, \tilde{g}(t), \tilde{\nabla}(t) \right)$, *where* $\tilde{\nabla}(t)$ *are torsion free connections on* $T\tilde{M}$, *is said to be a solution to the* ***Ricci flow for degenerate metrics*** *if* $\tilde{\nabla}\tilde{g} = 0$ *and*

$$\frac{\partial}{\partial t} \tilde{g}^{ab} = 2 \tilde{g}^{ac} \tilde{g}^{bd} \tilde{R}_{cd} \tag{9.3}$$

$$\frac{\partial}{\partial t} \tilde{\Gamma}_{ab}^{c} = -\tilde{g}^{cd} \left(\tilde{\nabla}_a \tilde{R}_{bd} + \tilde{\nabla}_b \tilde{R}_{ad} - \tilde{\nabla}_d \tilde{R}_{ab} \right) \tag{9.4}$$

where $\tilde{R}_{ab}$ *is the Ricci tensor of the connection* $\tilde{\nabla}$.

Note that equations (9.3)-(9.4) are the exact analogues of (9.1)-(9.2).

Let $(M^n, g(t))$ be a solution to the Ricci flow defined on a time interval $\mathcal{I} \subset \mathbb{R}$. We now consider the **space-time** manifold $\tilde{M}^{n+1} = M^n \times \mathcal{I}$. Define the **degenerate metric** $\tilde{g}$ **on the space-time cotangent bundle** $T\tilde{M}^*$ by

$$\tilde{g}(x,t) = g^{-1}(t)(x). \tag{9.5}$$

[1] We choose the manifold to have dimension $n+1$ simply becuase we have space-time in mind.

This metric is zero in the time direction. Let $x^0 = t$ be the time coordinate and again let $\{x^i\}_{i=1}^n$ be local coordinates on M^n. Then

$$\tilde{g}^{ij} = g^{ij}$$
$$\tilde{g}^{i0} = \tilde{g}^{0j} = 0$$
$$\tilde{g}^{00} = 0.$$

Presumably the degeneracy of the metric in the time direction is related to the infinite speed of propagation of the heat equation. For example, the forward light cones of the Lorentz metric $dx^2 - Ndt^2$ on $\mathbb{R}^2$ flatten out to the upper half plane as $N \to \infty$.

Define the **space-time connections** $\tilde{\nabla}(t)$ on $T\tilde{M}$ by

$$\tilde{\Gamma}_{ij}^k = \Gamma_{ij}^k \tag{9.6}$$
$$\tilde{\Gamma}_{i0}^k = \tilde{\Gamma}_{0i}^k = -R_i^k \tag{9.7}$$
$$\tilde{\Gamma}_{00}^k = -\frac{1}{2}\nabla^k R, \tag{9.8}$$

and all of the other components are zero.

NOTATION 9.3. *Here and throughout this chapter, $i, j, k, \ldots$ will run from 1 to n whereas $a, b, c, \ldots$ will run from 0 to n.*

The naturality of the above defined connections are exhibited by the following.

PROPOSITION 9.4 (Chow and S.-C. Chu 1995, [**140**]). *If $(M^n, g(t))$ is a solution of the Ricci flow, then the triple $\left(\tilde{M}^{n+1}, \tilde{g}(t), \tilde{\nabla}(t)\right)$ is a solution to the Ricci flow for degenerate metrics.*

First note that (9.6)-(9.7) imply

$$\tilde{\nabla}\tilde{g} = 0.$$

For example,

$$\tilde{\nabla}_0 \tilde{g}^{ij} = \frac{\partial}{\partial t} g^{ij} + \tilde{\Gamma}_{0k}^i g^{kj} + \tilde{\Gamma}_{0k}^j g^{ik} = 0.$$

EXERCISE 9.5 (Connections $\tilde{\nabla}$ satisfying the $\tilde{\nabla}\tilde{g} = 0$ condition). *Show that a torsion-free connection $\tilde{\nabla}$ on $T\tilde{M}$ is compatible with $\tilde{g}$ ($\tilde{\nabla}\tilde{g} = 0$) if and only if*

$$\tilde{\Gamma}_{ij}^k = \Gamma_{ij}^k$$
$$\tilde{\Gamma}_{i0}^k = -R_i^k + A_i^k$$
$$\tilde{\Gamma}_{ij}^0 = \tilde{\Gamma}_{0j}^0 = 0$$

for some 2-form A, and where $i, j, k \geq 1$.

The proof of the proposition requires the calculation of the space-time Ricci curvature tensor (see [**140**] for details). The **space-time Riemann curvature tensor** is defined by

$$\widetilde{\mathrm{Rm}}(X,Y)Z \doteqdot \tilde{\nabla}_X \tilde{\nabla}_Y Z - \tilde{\nabla}_Y \tilde{\nabla}_X Z - \tilde{\nabla}_{[X,Y]} Z,$$

and the **space-time Ricci curvature tensor** is defined by

$$\widetilde{\mathrm{Rc}}(Y,Z) \doteqdot \mathrm{trace}\left(X \mapsto \widetilde{\mathrm{Rm}}(X,Y)Z\right).$$

The second Bianchi identity says

$$\left(\tilde{\nabla}_X \widetilde{\mathrm{Rm}}\right)(Y,Z)W + \left(\tilde{\nabla}_Y \widetilde{\mathrm{Rm}}\right)(Z,X)W + \left(\tilde{\nabla}_Z \widetilde{\mathrm{Rm}}\right)(X,Y)W = 0. \tag{9.9}$$

LEMMA 9.6 (Space-time Ricci curvatures).

$$\tilde{R}_{jk} = R_{jk} \tag{9.10}$$

$$\tilde{R}_{0k} = \frac{1}{2}\nabla_k R \tag{9.11}$$

$$\tilde{R}_{00} = \frac{1}{2}\frac{\partial R}{\partial t} = \frac{1}{2}\Delta R + |\mathrm{Rc}|^2. \tag{9.12}$$

REMARK 9.7. *Note the identities*

$$\boxed{\tilde{R}_a^b = -\tilde{\Gamma}_{a0}^b} \tag{9.13}$$

where $a, b \geq 0$ *and* $\tilde{R}_a^b \doteqdot \tilde{g}^{bc}\tilde{R}_{ac}$.

The above lemma follows easily from tracing the formulas for the space-time Riemann curvature tensor. We omit the calculations since they are very similar to the ones carried out in the proof of Lemma 9.23 in section 3.

LEMMA 9.8 (Space-time Riemann curvatures).

$$\tilde{R}_{ijk}^{\ell} = R_{ijk}^{\ell} \tag{9.14}$$

$$\tilde{R}_{ij0}^{\ell} = -\nabla_i R_j^{\ell} + \nabla_j R_i^{\ell} \tag{9.15}$$

$$\tilde{R}_{i0k}^{\ell} = -\nabla^{\ell} R_{ik} + \nabla_k R_i^{\ell} \tag{9.16}$$

$$\begin{aligned} \tilde{R}_{i00}^{\ell} &= \Delta R_i^{\ell} - \frac{1}{2}\nabla_i \nabla^{\ell} R + 2g^{\ell m} R_{pim}^{q} R_q^p - R_m^{\ell} R_i^m \\ &= \frac{\partial}{\partial t} R_i^{\ell} - \frac{1}{2}\nabla_i \nabla^{\ell} R - R_i^m R_m^{\ell}, \end{aligned} \tag{9.17}$$

where all of the indices are ≥ 1, *and the rest of the components are zero.*

PROOF OF PROPOSITION 9.4. Equation (9.3) is trivial to verify (it basically follows from (9.1)), so we consider (9.4). We just make a sample calculation, where $a = i$, $b = 0$, and $c = k$. The LHS of (9.4) is

$$\frac{\partial}{\partial t}\tilde{\Gamma}_{i0}^k = -\frac{\partial}{\partial t}\left(g^{k\ell} R_{i\ell}\right).$$

Noting the identity

$$\tilde{\nabla}_i \tilde{R}_{0\ell} = \frac{1}{2}\nabla_i \nabla_\ell R + R_i^p R_{p\ell} = \tilde{\nabla}_\ell \tilde{R}_{i0},$$

the RHS of (9.4) may be written as

$$\begin{aligned} & -\tilde{g}^{k\ell}\left(\tilde{\nabla}_i \tilde{R}_{0\ell} + \tilde{\nabla}_0 \tilde{R}_{i\ell} - \tilde{\nabla}_\ell \tilde{R}_{i0}\right) \\ & = -g^{k\ell}\tilde{\nabla}_0 \tilde{R}_{i\ell} \\ & = -g^{k\ell}\left(\frac{\partial}{\partial t}R_{i\ell} - \tilde{\Gamma}_{0i}^p \tilde{R}_{p\ell} - \tilde{\Gamma}_{0\ell}^p \tilde{R}_{ip}\right) \\ & = -g^{k\ell}\left(\frac{\partial}{\partial t}R_{i\ell} + 2R_{ip}R_{p\ell}\right) \end{aligned}$$

and (9.4) follows in this case. We leave it to the reader to check the other cases such as the formula for $\frac{\partial}{\partial t}\tilde{\Gamma}_{00}^k$. □

A slight modification of $\tilde{\Gamma}$ is to define $\bar{\Gamma}$ by

$$\bar{\Gamma}_{ab}^c \doteqdot \tilde{\Gamma}_{ab}^c \tag{9.18}$$

except

$$\bar{\Gamma}_{00}^0 \doteqdot -\frac{1}{2t} \tag{9.19}$$

(see (B6) below for a motivation for this). The corresponding Riemann curvature tensor $\overline{\mathrm{Rm}}$ is then given by

$$\bar{R}_{ijk}^\ell \doteqdot R_{ijk}^\ell = \tilde{R}_{ijk}^\ell \tag{9.20}$$

$$\bar{R}_{ij0}^\ell \doteqdot -\nabla_i R_j^\ell + \nabla_j R_i^\ell = \tilde{R}_{ij0}^\ell \tag{9.21}$$

$$\begin{aligned} \bar{R}_{i00}^\ell & \doteqdot \Delta R_i^\ell - \frac{1}{2}\nabla_i \nabla^\ell R + 2g^{\ell m} R_{pim}^q R_q^p - R_m^\ell R_i^m + \frac{1}{2t}R_i^\ell \\ & = \tilde{R}_{i00}^\ell + \frac{1}{2t}R_i^\ell. \end{aligned} \tag{9.22}$$

The corresponding Ricci tensor is given by

$$\bar{R}_{jk} = R_{jk} \tag{9.23}$$

$$\bar{R}_{0k} = \frac{1}{2}\nabla_k R \tag{9.24}$$

$$\bar{R}_{00} = \frac{1}{2}\left(\frac{\partial R}{\partial t} + \frac{R}{t}\right). \tag{9.25}$$

We see that

$$\bar{R}_{i00}^\ell = M_i^\ell \quad \text{and} \quad \tilde{R}_{i00}^\ell = \bar{M}_i^\ell \tag{9.26}$$

as defined in (8.39) and (8.70);

$$g_{k\ell}\bar{R}_{ij0}^\ell = g_{k\ell}\tilde{R}_{ij0}^\ell = -P_{ijk} \tag{9.27}$$

as defined in (8.38). These are the components of Hamilton's trace and matrix Harnack quadratics. See section 2 for a further discussion of this.

REMARK 9.9. *Similar to Proposition 9.4, the triple* $\left(\tilde{M}^{n+1}, \tilde{g}(t), \bar{\nabla}(t)\right)$ *is almost a solution to the Ricci flow for degenerate metrics; the only case of (9.4) which does not hold is* $\frac{\partial}{\partial t}\bar{\Gamma}^0_{00} = \frac{1}{2t^2} \neq 0$.

EXERCISE 9.10 (see [**140**]). *Given a solution to Ricci flow on* $(0,T)$, *define the space-time Riemannian metric* ${}^N g(x,t) \doteqdot g(x,t) + \left(R + \frac{N}{2t}\right) dt^2$, *where* $N \in (0,\infty)$ *is large enough so that* $R + \frac{N}{2t} > 0$. *Show that (9.6)-(9.8) hold:*

$$\begin{aligned} {}^N\Gamma^k_{ij} &= \Gamma^k_{ij} \\ {}^N\Gamma^k_{i0} &= -R^k_i \\ {}^N\Gamma^k_{00} &= -\frac{1}{2}\nabla^k R \end{aligned}$$

and

$$\begin{aligned} {}^N\Gamma^0_{ij} &= \left(R + \frac{N}{2t}\right)^{-1} R_{ij} \\ {}^N\Gamma^0_{i0} &= \left(R + \frac{N}{2t}\right)^{-1} \frac{1}{2}\nabla_i R \\ {}^N\Gamma^0_{00} &= \left(R + \frac{N}{2t}\right)^{-1} \frac{1}{2}\left(\frac{\partial R}{\partial t} + \frac{R}{t}\right) - \frac{1}{2t}. \end{aligned}$$

Thus, taking $N \to \infty$, *we obtain the limit (9.18)-(9.19). For a generalization of this, see (B1)-(B9) below.*

An alternative way to get $\overline{\mathrm{Rm}}$ from the space-time approach is to add a **cosmological constant** to the Ricci flow equation and consider:

$$\frac{\partial}{\partial t} g_{ij} = -2\left(R_{ij} + \frac{1}{2} g_{ij}\right).$$

The reason for doing this is that *steady* Ricci solitons of the **Ricci flow with cosmological term** correspond to *expanding* Ricci solitons of the Ricci flow. In particular, if $(M^n, g(t))$ is a solution to the Ricci flow, then the metrics $\hat{g}_{ij}(t) \doteqdot \frac{1}{t} g(t)$, where $\hat{t} \doteqdot \log t$, satisfy $\frac{\partial}{\partial \hat{t}}\hat{g}_{ij} = -2\left(\hat{R}_{ij} + \frac{1}{2}\hat{g}_{ij}\right)$. If $R_{ij} + \nabla_i\nabla_j f + \frac{1}{2t} g_{ij} = 0$, then $\left(\hat{R}_{ij} + \frac{1}{2}\hat{g}_{ij}\right) + \hat{\nabla}_i\hat{\nabla}_j f = 0$. Defining the space-time connection for solutions of the Ricci flow with cosmological term appropriately (see Knopf and one of the authors [**152**]), we find that their curvatures are given by $\overline{\mathrm{Rm}}$ defined in (9.20)-(9.22), which again corresponds exactly to the matrix Harnack quadratic.

One has the following interesting identities for the exterior covariant derivative of the Ricci tensor as a $(1,1)$-tensor, i.e., as a 1-form with values in the tangent bundle (see Lemma 3.3 of [**140**]).

LEMMA 9.11 (Ricci soliton identities).

$$\begin{aligned}\tilde{R}_{i0k}^{\ell} &= \tilde{\nabla}_k \tilde{R}_i^{\ell} - \tilde{\nabla}^{\ell} \tilde{R}_{ik} \\ \tilde{R}_{i00}^{\ell} &= \tilde{\nabla}_0 \tilde{R}_i^{\ell} - \tilde{\nabla}^{\ell} \tilde{R}_{i0} \\ \tilde{R}_{00k}^{\ell} &= \tilde{\nabla}_k \tilde{R}_0^{\ell} - \tilde{\nabla}^{\ell} \tilde{R}_{0k} = 0 \\ \tilde{R}_{000}^{\ell} &= \tilde{\nabla}_0 \tilde{R}_0^{\ell} - \tilde{\nabla}^{\ell} \tilde{R}_{00} = 0\end{aligned}$$

where all of the indices are ≥ 1.

We may rewrite the above equations as

$$\tilde{\nabla}_c \tilde{R}_a^d - \tilde{\nabla}^d \tilde{R}_{ac} = \tilde{R}_{abc}^d \tilde{X}^b \tag{9.28}$$

where

$$\tilde{X} \doteqdot \frac{\partial}{\partial t}$$

($\tilde{X}^0 = 1$ and $\tilde{X}^i = 0$ for $i \geq 1$), and all of the indices are ≥ 0. Note that (9.28) is formally equivalent to (8.45), which follows from the Ricci soliton equation. Indeed, going back to the identity (9.13), we find that the **space-time Ricci soliton equation** is satisfied:

$$\boxed{\tilde{R}_a^b + \tilde{\nabla}_a \tilde{X}^b = \tilde{R}_a^b + \tilde{\Gamma}_{a0}^b \tilde{X}^0 = 0.} \tag{9.29}$$

See section 3 below and [**417**], §6 for related remarks. On the other hand, if we let $\tilde{f} = t$, then $\tilde{\nabla}_a \tilde{\nabla}_b \tilde{f} = -\tilde{\Gamma}_{ab}^0 \frac{\partial}{\partial t} \tilde{f} = 0$, so that we cannot quite write the space-time metric and connection pair as a *gradient* Ricci soliton.

Finally we leave it to the reader to verify that

LEMMA 9.12 (Divergence structure of $\widetilde{\mathrm{Rc}}$).

$$\tilde{g}^{ab} \tilde{\nabla}_a \tilde{R}_{bc} = g^{ij} \tilde{\nabla}_i \tilde{R}_{jc} = \tilde{R}_{0c}. \tag{9.30}$$

$$-\tilde{\nabla}_a \tilde{R}_b^d + \tilde{\nabla}_b \tilde{R}_a^d = \tilde{g}^{ef} \tilde{\nabla}_e \tilde{R}_{abf}^d = \tilde{R}_{ab0}^d. \tag{9.31}$$

We may rewrite (9.30) as

$$-\tilde{g}^{ab} \tilde{\nabla}_a \tilde{R}_{bc} + \tilde{R}_{dc} \tilde{X}^d = 0.$$

This is formally the same as the equation $-\frac{1}{2} \operatorname{div}(\mathrm{Rc}) + \mathrm{Rc}(X) = 0$, which holds on a Ricci soliton flowing along a vector field X (compare with (8.14)). Similarly, (9.31) may be rewritten as

$$\tilde{\nabla}_a \tilde{R}_b^d - \tilde{\nabla}_b \tilde{R}_a^d = -\tilde{R}_{abc}^d \tilde{X}^c$$

which we compare to (8.45).

EXERCISE 9.13. *Using the space-time second Bianchi identity, show that the equations in Lemma 9.11 exhibit a divergence structure for* $\widetilde{\mathrm{Rm}}$.

2. Space-time curvature is the matrix Harnack quadratic

Now we identify Hamilton's matrix Harnack quadratic with the Riemann curvature tensor of the space-time connection defined above. From (9.14), (8.40) and (8.41) ((9.27) and (9.26)) we have the following.

PROPOSITION 9.14 (Chow and S.-C. Chu 1995). *Let $\tilde{\nabla}$ be the connection on $T\tilde{M}$ defined by (9.6)-(9.8). Given a 2-form U and a 1-form W, if we define the space-time 2-form $\tilde{T}$ by $\tilde{T}_{ij} = U_{ij}$ and $\tilde{T}_{i0} = W_i$, then*

$$\operatorname{Rm}\left(\tilde{\nabla}\right)_{abcd} \tilde{T}_{ab}\tilde{T}_{dc} = Z\left(U, W\right) - \frac{1}{2t} R_{ij} W_i W_j.$$

We obtain a more precise identification by considering $\bar{\nabla}$:

$$\operatorname{Rm}\left(\bar{\nabla}\right)_{abcd} \tilde{T}_{ab}\tilde{T}_{dc} = Z\left(U, W\right).$$

Similarly, Hamilton's trace Harnack quadratic may essentially be identified with the Ricci tensor of the space-time connection. If V is a 1-form, define the space-time 1-form $\tilde{V} = V + dt$. Then

$$2\operatorname{Rc}\left(\tilde{\nabla}\right)_{ab} \tilde{V}_a\tilde{V}_b = \frac{\partial R}{\partial t} + 2\nabla R \cdot V + 2\operatorname{Rc}\left(V, V\right).$$

Combining the second Bianchi identity (9.9):

$$\tilde{\nabla}_0 \tilde{R}_{abc}^d = -\tilde{\nabla}_a \tilde{R}_{b0c}^d - \tilde{\nabla}_b \tilde{R}_{0ac}^d,$$

and the divergence identities $\tilde{R}_{b0c}^d = \tilde{g}^{ef}\tilde{\nabla}_e \tilde{R}_{bfc}^d$ and $\tilde{R}_{0ac}^d = \tilde{g}^{ef}\tilde{\nabla}_e \tilde{R}_{fac}^d$ yields a quick derivation of the evolution of the space-time Riemann curvature tensor (i.e., the matrix Harnack quadratic). We compute

$$\begin{aligned}
\tilde{\nabla}_0 \tilde{R}_{abc}^d &= -\tilde{g}^{ef}\tilde{\nabla}_a\tilde{\nabla}_e \tilde{R}_{bfc}^d + \tilde{g}^{ef}\tilde{\nabla}_b\tilde{\nabla}_e \tilde{R}_{afc}^d \\
&= -\tilde{g}^{ef}\left(\tilde{\nabla}_e\tilde{\nabla}_a \tilde{R}_{bfc}^d - \tilde{R}_{aeb}^h \tilde{R}_{hfc}^d - \tilde{R}_{aef}^h \tilde{R}_{bhc}^d - \tilde{R}_{aec}^h \tilde{R}_{bfh}^d + \tilde{R}_{aeh}^d \tilde{R}_{bfc}^h\right) \\
&\quad + \tilde{g}^{ef}\left(\tilde{\nabla}_e\tilde{\nabla}_b \tilde{R}_{afc}^d - \tilde{R}_{bea}^h \tilde{R}_{hfc}^d - \tilde{R}_{bef}^h \tilde{R}_{ahc}^d - \tilde{R}_{bec}^h \tilde{R}_{afh}^d + \tilde{R}_{beh}^d \tilde{R}_{afc}^h\right)
\end{aligned}$$

where the third line is minus the second line with a and b switched. Since $-\tilde{\nabla}_a \tilde{R}_{bfc}^d + \tilde{\nabla}_b \tilde{R}_{afc}^d = \tilde{\nabla}_f \tilde{R}_{abc}^d$, we have

LEMMA 9.15 (Evolution of space-time Riemann curvature tensor).

$$\begin{aligned}
\tilde{\nabla}_0 \tilde{R}_{abc}^d &= \tilde{g}^{ef}\tilde{\nabla}_e\tilde{\nabla}_f \tilde{R}_{abc}^d + \tilde{g}^{ef}\left(\tilde{R}_{aeb}^h \tilde{R}_{hfc}^d + \tilde{R}_{aec}^h \tilde{R}_{bfh}^d - \tilde{R}_{aeh}^d \tilde{R}_{bfc}^h\right) \\
&\quad + \tilde{g}^{ef}\left(-\tilde{R}_{bea}^h \tilde{R}_{hfc}^d - \tilde{R}_{bec}^h \tilde{R}_{afh}^d + \tilde{R}_{beh}^d \tilde{R}_{afc}^h\right) - \tilde{R}_a^h \tilde{R}_{hbc}^d - \tilde{R}_b^h \tilde{R}_{ahc}^d, \qquad (9.32)
\end{aligned}$$

where $a, b, c, d, e, f, h \geq 0$.

EXERCISE 9.16. *Show that if we use the connection $\bar{\nabla}$ defined by (9.18)-(9.19) (which by Exercise 9.10 is the adiabatic limit of Levi-Civita connections), then*

$$\operatorname{Rm}\left(\bar{\nabla}\right)\left(\tilde{T}, \tilde{T}\right) = Z\left(U, W\right)$$

and $\operatorname{Rc}\left(\tilde{V}, \tilde{V}\right)$ *is the trace Harnack (with the $\frac{R}{t}$ term.)*

EXERCISE 9.17. *Identify Hamilton's calculations of the evolution of the matrix Harnack quadratic in §6 of Chapter 8 with (9.32).*

3. Potentially infinite metrics and potentially infinite dimensions

3.1. The metric. The space-time connection is actually the limit of a 1-parameter family of Levi-Civita connections of metrics which are becoming infinite in the time direction. This idea of taking **potentially infinite metrics on space-time** first appeared in [**140**], [**141**]. In [**417**], §6 Perelman introduced two new ideas. First he dualized the above metric by introducing a minus sign in one of the terms. Second he took the product of the space-time manifold with a solution to the Ricci flow of **potentially infinite dimensions** (such as large radius and dimension spheres). More generally, one can consider the following. Let $(M^n, g(t))$ and $\left(P^N, a \cdot t \cdot h\right)$, $t \in \mathcal{I} \subset \mathbb{R} - \{0\}$, be solutions to Ricci flow, where $\left(P^N, h\right)$ is a fixed N-dimensional Riemannian Einstein manifold and $a \in \mathbb{R} - \{0\}$ with $a \cdot t > 0$ for all $t \in \mathcal{I}$. By the Ricci flow equation we have

$$\frac{\partial}{\partial t} g_{ij} = -2R_{ij}, \qquad \boxed{\operatorname{Rc}(h) = -\tfrac{a}{2} h.} \tag{9.33}$$

Given local coordinates $\left\{x^i\right\}_{i=1}^n$ on M^n and $\{y^\alpha\}_{a=1}^N$ on P^N we write $g_{ij}(t) = g(t)\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right)$ and $h_{\alpha\beta} = h\left(\frac{\partial}{\partial y^\alpha}, \frac{\partial}{\partial y^\beta}\right)$. Let $\Delta_{\alpha\beta}^\gamma$ denote the Christoffel symbols of h. Define the Riemannian metric ${}^N g(t)$ on $M^n \times \mathcal{I} \times P^N$ by

$$\boxed{{}^N g(t) = g(t) + \left(R(g(t)) + \tfrac{bN}{2t}\right) dt^2 + ath,} \tag{9.34}$$

where $a, b, t \neq 0$. Here we have assumed

$$R(g(t)) \geq -C/|t| \tag{9.35}$$

for some $C < \infty$, and that bN is large enough, and $bt > 0$, so the metrics ${}^N g(t)$ are positive definite.

Note that $R(ath) = \frac{bN}{2t}$ if and only if $b = -1$. That is,

$${}^N g(t) = g(t) + R(g(t) + ath)\, dt^2 + ath$$

if and only if $b = -1$.

REMARK 9.18. *As long as for each time t, $R(g(t))$ is bounded from below and provided we can apply the maximum principle, condition (9.35) holds. For then, if $\mathcal{I} = [\alpha, 0)$ and the scalar curvature is bounded below at time α, then R is uniformly bounded from below on $\mathcal{I}$. On the other hand, if $\mathcal{I} = (0, \omega)$, then from $\frac{\partial}{\partial t} R \geq \Delta R + \frac{2}{n} R^2$, we conclude $R(g(t)) \geq -\frac{n}{2t}$.*

In local coordinates, the space-time metric ${}^N g$ is:

$$ {}^N g_{ij} = g_{ij} \tag{A1} $$

$$ {}^N g_{00} = R + \frac{bN}{2t} \tag{A2} $$

$$ {}^N g_{\alpha\beta} = a t h_{\alpha\beta} \tag{A3} $$

$$ {}^N g_{i0} = {}^N g_{\alpha 0} = {}^N g_{i\alpha} = 0. \tag{A4} $$

3.2. The connection. Recall the standard formula for the Christoffel symbols:

$$ {}^N\Gamma^c_{ab} = \frac{1}{2}\, {}^N g^{cd} \left(\frac{\partial}{\partial z^a}\, {}^N g_{bd} + \frac{\partial}{\partial z^b}\, {}^N g_{ad} - \frac{\partial}{\partial z^d}\, {}^N g_{ab} \right), \tag{9.36} $$

(where $a, b, c = 0, i, j, k..., \alpha, \beta, \gamma, ...$ and $z = x$ or y). A straightforward calculation yields the following result.

LEMMA 9.19 (Space-time Christoffel symbols of potentially infinite metric). *Suppose $g(t)$ and h satisfy (9.33), N is large enough, and $a, b, t \neq 0$ have the same sign (in the following, it will be obvious when a and b are constants and when they are indices). Using formula (9.36) we compute that the Christoffel symbols of ${}^N g$ are given by:*

$$ {}^N\Gamma^k_{ij} = \Gamma^k_{ij} \tag{B1} $$

$$ {}^N\Gamma^k_{i0} = -R^k_i \tag{B2} $$

$$ {}^N\Gamma^k_{00} = -\frac{1}{2}\nabla^k R \tag{B3} $$

$$ {}^N\Gamma^0_{ij} = \left(R + \frac{bN}{2t} \right)^{-1} R_{ij} \tag{B4} $$

$$ {}^N\Gamma^0_{i0} = \left(R + \frac{bN}{2t} \right)^{-1} \frac{1}{2}\nabla_i R \tag{B5} $$

$$ {}^N\Gamma^0_{00} = \left(R + \frac{bN}{2t} \right)^{-1} \frac{1}{2}\left(\frac{\partial R}{\partial t} + \frac{R}{t} \right) - \frac{1}{2t} \tag{B6} $$

$$ {}^N\Gamma^\gamma_{\alpha\beta} = \Delta^\gamma_{\alpha\beta} \tag{B7} $$

$$ {}^N\Gamma^\gamma_{\alpha 0} = \frac{1}{2t}\delta^\gamma_\alpha \tag{B8} $$

$$ {}^N\Gamma^0_{\alpha\beta} = -\frac{1}{2}\left(R + \frac{bN}{2t} \right)^{-1} a h_{\alpha\beta} \tag{B9} $$

and the rest of the components are 0.

CAVEAT. 0 denotes the t component, not the τ component. This causes a sign difference in the terms of the Christoffel symbol with an odd number of 0 indices as compared to some other expositions of Perelman's space-time metric.

REMARK 9.20.

(1) *Equations (B1)-(B3) for* ${}^N\Gamma_{ab}^k$ *are the same as (9.6)-(9.8) for* $\tilde{\Gamma}_{ab}^k$*, where* $a, b = 0, 1, \ldots, n$.

(2) *Note*

$$
\begin{aligned}
{}^N\Gamma_{a0}^k &= -\tilde{R}_a^k = -\bar{R}_a^k \\
{}^N\Gamma_{ab}^0 &= \left(R + \frac{bN}{2t}\right)^{-1} \bar{R}_{ab} - \frac{1}{2t}\delta_{a0}\delta_{b0}
\end{aligned}
$$

for $a, b = 0, 1, \ldots, n$ *and* $k = 1, , \ldots, n$*, and where* $\bar{R}_{ab}$ *is the space-time symmetric 2-tensor which represents Hamilton's trace Harnack quadratic defined by (9.23)-(9.25).*

(3) *We have the following symmetries:*

$$
{}^N\Gamma_{ia}^0 = -\ {}^N g^{00}\ {}^N g_{ik}\ {}^N\Gamma_{a0}^k,
$$

where $a = 0, 1, \ldots, n$*, which relates (B2) to (B4) and (B3) to (B5);*

$$
{}^N\Gamma_{\alpha\beta}^0 = -\ {}^N g^{00}\ {}^N g_{\alpha\gamma}\ {}^N\Gamma_{\beta 0}^\gamma
$$

which relates (B8) to (B9).

PROOF. We make a few sample calculations (B3):

$$
\begin{aligned}
{}^N\Gamma_{00}^k &= \frac{1}{2}\ {}^N g^{k\ell}\left(2\frac{\partial}{\partial z^0}\ {}^N g_{0\ell} - \frac{\partial}{\partial z^\ell}\ {}^N g_{00}\right) \\
&= -\frac{1}{2} g^{k\ell}\frac{\partial}{\partial x^\ell}\left(R + \frac{bN}{2t}\right)
\end{aligned}
$$

and (B5):

$$
\begin{aligned}
{}^N\Gamma_{i0}^0 &= \frac{1}{2}\ {}^N g^{00}\left(\frac{\partial}{\partial z^i}\ {}^N g_{00} + \frac{\partial}{\partial z^0}\ {}^N g_{i0} - \frac{\partial}{\partial z^0}\ {}^N g_{i0}\right) \\
&= \frac{1}{2}\left(R + \frac{bN}{2t}\right)^{-1}\frac{\partial}{\partial x^i}\left(R + \frac{bN}{2t}\right)
\end{aligned}
$$

and (B8):

$$
{}^N\Gamma_{\alpha 0}^\gamma = \frac{1}{2}\ {}^N g^{\gamma\delta}\left(\frac{\partial}{\partial y^\alpha}\ {}^N g_{0\delta} + \frac{\partial}{\partial x^0}\ {}^N g_{\alpha\delta} - \frac{\partial}{\partial z^\delta}\ {}^N g_{\alpha 0}\right) = \frac{1}{2t}\ \delta_\alpha^\gamma,
$$

and leave the rest of the calculations to the reader. □

REMARK 9.21. *Note that if* $a = 0$*, then* ${}^N\Gamma_{\alpha 0}^\gamma = 0$*. To obtain the desired cancellations in Theorem 9.31 it is essential that* $a \neq 0$*. In particular, when* $a = 0$*, formula (C9) becomes* ${}^N R_{\alpha 00}^\delta = 0$ *(and hence* $\sum_\alpha {}^N R_{\alpha 00}^\alpha$ *does not yield a contribution to* ${}^N R_{00}$*.)*

3.3. The Laplacian. Consider the potentially infinite metrics ${}^N g$ given by (A1)-(A4) on the product of space-time with the potentially infinite dimensional Einstein solutions: $M^n \times \mathcal{I} \times P^N$. The Laplacians of ${}^N g$ acting on functions constant on the P^N factor are given by:

$$\begin{aligned} {}^N\Delta &= {}^N g^{ab} \left(\frac{\partial^2}{\partial x^a \partial x^b} - {}^N\Gamma^c_{ab} \frac{\partial}{\partial x^c} \right) \\ &= \Delta - \left({}^N g^{\alpha\beta}\, {}^N\Gamma^0_{\alpha\beta} + {}^N g^{ij}\, {}^N\Gamma^0_{ij} + {}^N g^{00}\, {}^N\Gamma^0_{00} \right) \frac{\partial}{\partial t} \\ &+ {}^N g^{00} \frac{\partial^2}{\partial t^2} - {}^N g^{00}\, {}^N\Gamma^k_{00} \frac{\partial}{\partial x^k}, \end{aligned}$$

where Δ denotes the Laplacian with respect to g. In particular, using the formulas for the Christoffel symbols given by Lemma 9.19, we find that if $b \neq 0$, then

$$\boxed{{}^N\Delta = \Delta + b^{-1} \frac{\partial}{\partial t} + O\left(N^{-1}\right).} \tag{9.37}$$

For example, if $b = -1$, which is the case we are most interested in, then the space-time Laplacian is potentially the heat operator:[2]

$$\square \doteqdot \Delta - \frac{\partial}{\partial t}.$$

If $b = 1$, then we get the backward heat operator in the limit: $\Delta + \frac{\partial}{\partial t}$. Here, to get the heat operator as a limit, one sees from the calculation that the idea of taking the product with a potentially infinite dimensional Einstein solution P^N is crucial; for tracing an $O\left(N^{-1}\right)$ 2-tensor over a manifold with N dimensions yields an $O(1)$ term. As we shall see later, this idea is also crucial to obtain a potentially Ricci flat solution.

EXERCISE 9.22. *Show that*

$$\begin{aligned} {}^N\Delta = \Delta &+ \left[\left(1 + \frac{1 - 2tR}{N}\right) \left(b + \frac{2tR}{N}\right)^{-1} + O\left(N^{-2}\right) \right] \frac{\partial}{\partial t} \\ &+ \left(R + \frac{bN}{2t}\right)^{-1} \frac{\partial^2}{\partial t^2} + \left(R + \frac{bN}{2t}\right)^{-1} \frac{1}{2} \nabla R \cdot \nabla. \end{aligned}$$

Derive (9.37) from this.

The volume form of the metric ${}^N g(t)$ is given by (now take $a = 1$)

$$\begin{aligned} {}^N d\mu &= t^{N/2} \sqrt{R + \frac{bN}{2t}}\, d\mu_g \wedge dt \wedge d\mu_h \\ &= \sqrt{b/2} \left(N^{1/2} + O\left(N^{-1/2}\right)\right) t^{(N-1)/2} d\mu_g \wedge dt \wedge d\mu_h. \end{aligned}$$

[2]Perelman [**417**], §6.1.

Given a function u on $M^n \times \mathcal{I} \times P^N$ which is independent of the P^N variables,[3]

$$\begin{aligned}\tau^{\frac{N-1}{2}} \cdot {}^N\Delta\left(\tau^{-\frac{N-1}{2}} u\right) &= \left(\Delta + b^{-1}\frac{\partial}{\partial t}\right) u - \frac{N-1}{2t}\left(b + \frac{2tR}{N}\right)^{-1}\left(\frac{N+1}{N} - \frac{2tR}{N}\right) u \\ &\quad + \frac{N-1}{N}\left(b + \frac{2tR}{N}\right)^{-1}\left(-2\frac{\partial}{\partial t} + \frac{N+1}{2t} u\right) + O\left(N^{-1}\right) \\ &= \left(\Delta - b^{-1}\frac{\partial}{\partial t} + b^{-1} R\right) u + O\left(N^{-1}\right).\end{aligned}$$

Note that $\Delta - b^{-1}\frac{\partial}{\partial t} + b^{-1}R$ is the adjoint of the heat operator $\Delta + b^{-1}\frac{\partial}{\partial t}$ (see (**??**) for the case where $b = -1$.)

3.4. The Riemann curvature. Recall the components of the Riemann curvature tensor are given by:

$$ {}^N R_{abc}^d = \partial_a {}^N\Gamma_{bc}^d - \partial_b {}^N\Gamma_{ac}^d + {}^N\Gamma_{bc}^e {}^N\Gamma_{ae}^d - {}^N\Gamma_{ac}^e {}^N\Gamma_{be}^d. \tag{9.38}$$

Again a straightforward calculation yields (this time with a little more patience):

LEMMA 9.23 (Space-time Riemann curvature of potentially infinite metric). *Using formula (9.38) we compute that the nonzero components of the Riemann curvature tensor* ${}^N\mathrm{Rm}$ *of* ${}^N g$ *are given by*

$$ \text{(C1)}\quad {}^N R_{ijk}^{\ell} = \left[R_{ijk}^{\ell}\right] + \left(R + \frac{bN}{2t}\right)^{-1}\left(R_{ik} R_j^{\ell} - R_i^{\ell} R_{jk}\right)$$

$$ \text{(C2)}\quad {}^N R_{i0k}^{\ell} = \left[\nabla_k R_i^{\ell} - \nabla^{\ell} R_{ik}\right] + \left(R + \frac{bN}{2t}\right)^{-1}\left(R_{ik}\frac{1}{2}\nabla^{\ell} R - R_i^{\ell}\frac{1}{2}\nabla_k R\right)$$

$$\begin{aligned}\text{(C3)}\quad {}^N R_{i00}^{\ell} &= \left[\frac{\partial}{\partial t} R_i^{\ell} - \frac{1}{2}\nabla_i\nabla^{\ell} R - R_i^p R_p^{\ell} + \frac{1}{2t} R_i^{\ell}\right] \\ &\quad + \left(R + \frac{bN}{2t}\right)^{-1}\left(\frac{1}{2}\nabla_i R \frac{1}{2}\nabla^{\ell} R - R_i^{\ell}\frac{1}{2}\left(\frac{\partial R}{\partial t} + \frac{R}{t}\right)\right)\end{aligned}$$

[3] Again, Perelman [**417**], §6.1.

$$\text{(C4)}\qquad {}^{N}R^{0}_{ijk} = \left(R + \frac{bN}{2t}\right)^{-1}\left[\nabla_i R_{jk} - \nabla_j R_{ik}\right] + \left(R + \frac{bN}{2t}\right)^{-2}\left(R_{ik}\frac{1}{2}\nabla_j R - R_{jk}\frac{1}{2}\nabla_i R\right)$$

$$\text{(C5)}\qquad {}^{N}R^{0}_{i0k} = -\left(R + \frac{bN}{2t}\right)^{-1}\left[\frac{\partial}{\partial t}R_{ik} - \frac{1}{2}\nabla_i\nabla_k R + R^{p}_{k}R_{ip} + \frac{1}{2t}R_{ik}\right] + \left(R + \frac{bN}{2t}\right)^{-2}\left(R_{ik}\frac{1}{2}\left(\frac{\partial R}{\partial t} + \frac{R}{t}\right) - \frac{1}{2}\nabla_k R\frac{1}{2}\nabla_i R\right)$$

$$\text{(C6)}\qquad {}^{N}R^{\delta}_{\alpha\beta\gamma} = R^{\delta}_{\alpha\beta\gamma} + \frac{1}{2t}\left(R + \frac{bN}{2t}\right)^{-1}\frac{a}{2}\left(h_{\alpha\gamma}\delta^{\delta}_{\beta} - h_{\beta\gamma}\delta^{\delta}_{\alpha}\right) = R^{\delta}_{\alpha\beta\gamma} + \frac{1}{2t}\left(R + \frac{bN}{2t}\right)^{-1}\left(\operatorname{Rc}(h)_{\beta\gamma}\delta^{\delta}_{\alpha} - \operatorname{Rc}(h)_{\alpha\gamma}\delta^{\delta}_{\beta}\right)$$

$$\text{(C7)}\qquad {}^{N}R^{\delta}_{\alpha jk} = \frac{1}{2t}\delta^{\delta}_{\alpha}\left(R + \frac{bN}{2t}\right)^{-1}R_{jk}$$

$$\text{(C8)}\qquad {}^{N}R^{\delta}_{\alpha 0k} = \frac{1}{2t}\delta^{\delta}_{\alpha}\left(R + \frac{bN}{2t}\right)^{-1}\frac{1}{2}\nabla_k R$$

$$\text{(C9)}\qquad {}^{N}R^{\delta}_{\alpha 00} = \frac{1}{2t}\delta^{\delta}_{\alpha}\left(R + \frac{bN}{2t}\right)^{-1}\frac{1}{2}\left(\frac{\partial R}{\partial t} + \frac{R}{t}\right).$$

REMARK 9.24. *In the first five equations we have emphasized the terms on the* RHS *which appear in the matrix Harnack quadratic by putting them inside square brackets:* [ ]*. Note also the appearance on the* RHS *of terms in the trace Harnack quadratic.*

PROOF. Again, we only make some sample calculations:

$$\begin{aligned}{}^{N}R^{\ell}_{i00} &= \partial_i {}^{N}\Gamma^{\ell}_{00} - \partial_0 {}^{N}\Gamma^{\ell}_{i0} + {}^{N}\Gamma^{k}_{00}\,{}^{N}\Gamma^{\ell}_{ik} + {}^{N}\Gamma^{0}_{00}\,{}^{N}\Gamma^{\ell}_{i0} - {}^{N}\Gamma^{k}_{i0}\,{}^{N}\Gamma^{\ell}_{0k} - {}^{N}\Gamma^{0}_{i0}\,{}^{N}\Gamma^{\ell}_{00}\\ &= -\frac{1}{2}\nabla_i\nabla^{\ell}R + \frac{\partial}{\partial t}R^{\ell}_{i} - \left(R + \frac{bN}{2t}\right)^{-1}\frac{1}{2}\left(\frac{\partial R}{\partial t} - \frac{bN}{2t^2}\right)R^{\ell}_{i}\\ &\quad - R^{k}_{i}R^{\ell}_{k} + \left(R + \frac{bN}{2t}\right)^{-1}\frac{1}{2}\nabla_i R\frac{1}{2}\nabla^{\ell}R\end{aligned}$$

and

$$\begin{aligned}{}^{N}R^{0}_{ijk} &= \partial_i {}^{N}\Gamma^{0}_{jk} - \partial_j {}^{N}\Gamma^{0}_{ik} + {}^{N}\Gamma^{\ell}_{jk}\,{}^{N}\Gamma^{0}_{i\ell} + {}^{N}\Gamma^{0}_{jk}\,{}^{N}\Gamma^{0}_{i0} - {}^{N}\Gamma^{\ell}_{ik}\,{}^{N}\Gamma^{0}_{j\ell} - {}^{N}\Gamma^{0}_{ik}\,{}^{N}\Gamma^{0}_{j0}\\ &= \nabla_i\left(\left(R + \frac{bN}{2t}\right)^{-1}R_{jk}\right) - \nabla_j\left(\left(R + \frac{bN}{2t}\right)^{-1}R_{ik}\right)\\ &\quad + \left(R + \frac{bN}{2t}\right)^{-1}R_{jk}\left(R + \frac{bN}{2t}\right)^{-1}\frac{1}{2}\nabla_i R\\ &\quad - \left(R + \frac{bN}{2t}\right)^{-1}R_{ik}\left(R + \frac{bN}{2t}\right)^{-1}\frac{1}{2}\nabla_j R\end{aligned}$$

and

$$\begin{aligned}
{}^N R^{\delta}_{\alpha 00} &= \partial_\alpha {}^N\Gamma^{\delta}_{00} - \partial_0 {}^N\Gamma^{\delta}_{\alpha 0} + {}^N\Gamma^{\gamma}_{00} {}^N\Gamma^{\delta}_{\alpha\gamma} + {}^N\Gamma^{0}_{00} {}^N\Gamma^{\delta}_{\alpha 0} - {}^N\Gamma^{\gamma}_{\alpha 0} {}^N\Gamma^{\delta}_{0\gamma} - {}^N\Gamma^{0}_{\alpha 0} {}^N\Gamma^{\delta}_{00} \\
&= -\frac{\partial}{\partial t}\left(\frac{1}{2t}\right)\delta^{\delta}_{\alpha} + \left(R + \frac{bN}{2t}\right)^{-1} \frac{1}{2}\left(\frac{\partial R}{\partial t} - \frac{bN}{2t^2}\right)\frac{1}{2t}\delta^{\delta}_{\alpha} - \left(\frac{1}{2t}\right)^2 \delta^{\gamma}_{\alpha}\delta^{\delta}_{\gamma} \\
&= \frac{1}{2t}\delta^{\delta}_{\alpha}\left(R + \frac{bN}{2t}\right)^{-1} \frac{1}{2}\left(\frac{\partial R}{\partial t} + \frac{R}{t}\right)
\end{aligned}$$

and

$$ {}^N R^{\delta}_{\alpha jk} = {}^N\Gamma^{0}_{jk} {}^N\Gamma^{\delta}_{\alpha 0} = \frac{1}{2t}\delta^{\delta}_{\alpha}\left(R + \frac{bN}{2t}\right)^{-1} R_{jk}. $$

□

REMARK 9.25. *Note that*

$$\begin{aligned}
{}^N R^{0}_{ibk} &= -\left(R + \frac{bN}{2t}\right)^{-1} \bar{R}^{\ell}_{ib0} g_{\ell k} + \left(R + \frac{bN}{2t}\right)^{-2} \left(\bar{R}_{ik}\bar{R}_{b0} - \bar{R}_{bk}\bar{R}_{i0}\right) \\
{}^N R^{\ell}_{ibc} &= \bar{R}^{\ell}_{ibc} + \left(R + \frac{bN}{2t}\right)^{-1} \left(\bar{R}_{ic}\bar{R}^{\ell}_{b} - \bar{R}^{\ell}_{i}\bar{R}_{bc}\right) \\
{}^N R^{\delta}_{\alpha ab} &= \frac{1}{2t}\delta^{\delta}_{\alpha}\left(R + \frac{bN}{2t}\right)^{-1} \bar{R}_{ab}.
\end{aligned}$$

If we are only interested in the values of the curvature up to $O\left(N^{-1}\right)$, then we obtain the following.

COROLLARY 9.26 (Space-time Riemann up to $O\left(N^{-1}\right)$).

$$\begin{aligned}
{}^N R^{\ell}_{ijk} &= R^{\ell}_{ijk} + O\left(N^{-1}\right) \\
{}^N R^{\ell}_{i0k} &= \nabla_k R^{\ell}_{i} - \nabla^{\ell} R_{ik} + O\left(N^{-1}\right) \\
{}^N R^{\ell}_{i00} &= \frac{\partial}{\partial t} R^{\ell}_{i} - \frac{1}{2}\nabla_i\nabla^{\ell} R - R^{p}_{i} R^{\ell}_{p} + \frac{1}{2t} R^{\ell}_{i} + O\left(N^{-1}\right) \\
{}^N R^{\delta}_{\alpha\beta\gamma} &= R^{\delta}_{\alpha\beta\gamma} + O\left(N^{-1}\right)
\end{aligned}$$

and the rest of the terms are $O\left(N^{-1}\right)$.

It is interesting to note that up to $O\left(N^{-1}\right)$, the Riemann curvature $(3,1)$-tensor does not depend on a or b. This is somewhat misleading since the order of ${}^N\mathrm{Rm}$ is measured with respect to ${}^N g$, which is becoming infinite as N tends to infinity.

3.5. The Ricci curvature. The components of the Ricci tensor are given by the following.

PROPOSITION 9.27. *The components of* ${}^N\mathrm{Rc}$ *are:*

$$
\begin{aligned}
{}^N R_{jk} &= \left(R + \frac{bN}{2t}\right)^{-1} \left(R_j^\ell R_{\ell k} + (1+b)\,\frac{N}{2t} R_{jk}\right) \\
&\quad + \left(R + \frac{bN}{2t}\right)^{-1} \left[\frac{\partial}{\partial t} R_{jk} - \frac{1}{2}\nabla_j \nabla_k R + R_k^p R_{jp} + \frac{1}{2t} R_{jk}\right] \qquad (9.39) \\
&\quad - \left(R + \frac{bN}{2t}\right)^{-2} \left(R_{jk}\frac{1}{2}\left(\frac{\partial R}{\partial t} + \frac{R}{t}\right) - \frac{1}{2}\nabla_k R \frac{1}{2}\nabla_j R\right)
\end{aligned}
$$

$$
{}^N R_{00} = \left(R + \frac{bN}{2t}\right)^{-1} \left(\frac{1}{4}\left|\nabla R\right|^2 + (1+b)\,\frac{N}{2t}\frac{1}{2}\left(\frac{\partial R}{\partial t} + \frac{R}{t}\right)\right) \qquad (9.40)
$$

$$
{}^N R_{0k} = \left(R + \frac{bN}{2t}\right)^{-1} \left(R_{ik}\frac{1}{2}\nabla^i R + (b+1)\,\frac{N}{2t}\frac{1}{2}\nabla_k R\right) \qquad (9.41)
$$

$$
\begin{aligned}
{}^N R_{\beta\gamma} &= -\frac{a}{2}\left(R + \frac{bN}{2t}\right)^{-1} \left((b+1)\,N - 1\right)\frac{1}{2t} h_{\beta\gamma} \\
&\quad + \left(R + \frac{bN}{2t}\right)^{-2} \frac{a}{2} h_{\beta\gamma}\frac{1}{2}\left(\frac{\partial R}{\partial t} + \frac{R}{t}\right) \qquad (9.42)
\end{aligned}
$$

REMARK 9.28. *If* $b = -1$, *then*

$$
\begin{aligned}
{}^N R_{jk} &= \left(R - \frac{N}{2t}\right)^{-1} R_j^\ell R_{\ell k} \\
&\quad + \left(R - \frac{N}{2t}\right)^{-1} \left[\frac{\partial}{\partial t} R_{jk} - \frac{1}{2}\nabla_j \nabla_k R + R_k^p R_{jp} + \frac{1}{2t} R_{jk}\right] \\
&\quad - \left(R - \frac{N}{2t}\right)^{-2} \left(R_{jk}\frac{1}{2}\left(\frac{\partial R}{\partial t} + \frac{R}{t}\right) - \frac{1}{2}\nabla_k R \frac{1}{2}\nabla_j R\right) \\
{}^N R_{00} &= \left(R - \frac{N}{2t}\right)^{-1} \frac{1}{4}\left|\nabla R\right|^2 \\
{}^N R_{0k} &= \left(R - \frac{N}{2t}\right)^{-1} \frac{1}{2} R_{ik} \nabla^i R \\
{}^N R_{\beta\gamma} &= \frac{a}{2}\left(R - \frac{N}{2t}\right)^{-1} \frac{1}{2t} h_{\beta\gamma} + \left(R - \frac{N}{2t}\right)^{-2} \frac{a}{2} h_{\beta\gamma}\frac{1}{2}\left(\frac{\partial R}{\partial t} + \frac{R}{t}\right)
\end{aligned}
$$

Now let $N^m = M^n \times P^N$.

COROLLARY 9.29 (Space-time Ricci up to $O\left(N^{-1}\right)$). *If* $g\left(t\right)$ *and* h *satisfy (9.33), then the nonzero components of the space time Ricci tensor* ${}^N\mathrm{Rc}$

are:

$$ {}^{N}R_{jk} = \left(1+\frac{1}{b}\right) R_{jk} + O\left(N^{-1}\right) \tag{9.43} $$

$$ {}^{N}R_{0k} = \left(1+\frac{1}{b}\right) \frac{1}{2}\nabla_k R + O\left(N^{-1}\right) \tag{9.44} $$

$$ {}^{N}R_{00} = \left(1+\frac{1}{b}\right) \frac{1}{2}\left(\frac{\partial R}{\partial t}+\frac{R}{t}\right) + O\left(N^{-1}\right) \tag{9.45} $$

$$ {}^{N}R_{\beta\gamma} = -\frac{a}{2}\left(1+\frac{1}{b}\right) h_{\beta\gamma} + O\left(N^{-1}\right). \tag{9.46} $$

PROOF. We derive the formulas for the components of the Ricci tensor using:

$$ {}^{N}R_{bc} = \sum_i {}^{N}R^{i}_{ibc} + {}^{N}R^{0}_{0bc} + \sum_\alpha {}^{N}R^{\alpha}_{\alpha bc} $$

and the formulas for the traces of the (nonzero) components of the Riemann curvature tensor, which are as follows. To find ${}^{N}R_{ij}$ we compute:

$$ \sum_i {}^{N}R^{i}_{ijk} = R_{jk} + \left(R+\frac{bN}{2t}\right)^{-1}\left(R_{ik}R^{i}_{j} - RR_{jk}\right) \tag{9.47} $$

$$ \begin{aligned} {}^{N}R^{0}_{0jk} = {} & \left(R+\frac{bN}{2t}\right)^{-1}\left[\frac{\partial}{\partial t}R_{jk} - \frac{1}{2}\nabla_j\nabla_k R + R^{p}_{k}R_{jp} + \frac{1}{2t}R_{jk}\right] \\ & - \left(R+\frac{bN}{2t}\right)^{-2}\left(R_{jk}\frac{1}{2}\left(\frac{\partial R}{\partial t}+\frac{R}{t}\right) - \frac{1}{2}\nabla_k R\frac{1}{2}\nabla_j R\right) \end{aligned} \tag{9.48} $$

$$ \sum_\alpha {}^{N}R^{\alpha}_{\alpha jk} = \frac{N}{2t}\left(R+\frac{bN}{2t}\right)^{-1} R_{jk}, \tag{9.49} $$

which imply (9.39). Next we consider ${}^{N}R_{0k}$:

$$ \sum_i {}^{N}R^{i}_{i0k} = \frac{1}{2}\nabla_k R + \left(R+\frac{bN}{2t}\right)^{-1}\left(R_{ik}\frac{1}{2}\nabla^i R - R\frac{1}{2}\nabla_k R\right) $$

$$ \sum_\alpha {}^{N}R^{\alpha}_{\alpha 0k} = \frac{N}{2t}\left(R+\frac{bN}{2t}\right)^{-1}\frac{1}{2}\nabla_k R $$

which imply (9.41). Third, we consider ${}^{N}R_{00}$:

$$ \sum_i {}^{N}R^{i}_{i00} = \frac{1}{2}\left(\frac{\partial R}{\partial t}+\frac{R}{t}\right) + \left(R+\frac{bN}{2t}\right)^{-1}\left(\frac{1}{4}\left|\nabla R\right|^2 - \frac{1}{2}R\left(\frac{\partial R}{\partial t}+\frac{R}{t}\right)\right) $$

$$ \sum_\alpha {}^{N}R^{\alpha}_{\alpha 00} = \left(b+\frac{2tR}{N}\right)^{-1}\frac{1}{2}\left(\frac{\partial R}{\partial t}+\frac{R}{t}\right) $$

which imply (9.40). Finally we consider ${}^N R_{\beta\gamma}$:

$$\sum_i {}^N R^i_{i\beta\gamma} = \frac{a}{2} R \left(R + \frac{bN}{2t} \right)^{-1} h_{\beta\gamma}$$

$$\begin{aligned} \sum_\alpha {}^N R^\alpha_{\alpha\beta\gamma} &= \mathrm{Rc}\,(h)_{\beta\gamma} - \frac{a}{4t} (N-1) \left(R + \frac{bN}{2t} \right)^{-1} h_{\beta\gamma} \\ &= -\frac{a}{2} \left(R - \frac{1}{2t} + (1+b)\frac{N}{2t} \right) \left(R + \frac{bN}{2t} \right)^{-1} h_{\beta\gamma}, \end{aligned}$$

$$\begin{aligned} {}^N R^0_{0\beta\gamma} &= {}^N g^{00}\, {}^N g_{\gamma\delta}\, {}^N R^\delta_{\beta 00} \\ &= \left(R + \frac{bN}{2t} \right)^{-2} \frac{a}{2} h_{\beta\gamma} \frac{1}{2} \left(\frac{\partial R}{\partial t} + \frac{R}{t} \right) \end{aligned}$$

which imply (9.42):

$$\begin{aligned} {}^N R_{\beta\gamma} &= \sum_i {}^N R^i_{i\beta\gamma} + \sum_\alpha {}^N R^\alpha_{\alpha\beta\gamma} + {}^N R^0_{0\beta\gamma} \\ &= -\frac{a}{2} \left(-\frac{1}{2t} + (1+b)\frac{N}{2t} \right) \left(R + \frac{bN}{2t} \right)^{-1} h_{\beta\gamma} \\ &+ \left(R + \frac{bN}{2t} \right)^{-2} \frac{a}{2} h_{\beta\gamma} \frac{1}{2} \left(\frac{\partial R}{\partial t} + \frac{R}{t} \right) \end{aligned}$$

□

Observe that the metrics Perelman considered are as follows:

$$\begin{aligned} {}^N \tilde{g}_{ij} &= g_{ij} \\ {}^N \tilde{g}_{\alpha\beta} &= \tau g_{\alpha\beta} \\ {}^N \tilde{g}_{00} &= R + \frac{N}{2\tau} \\ {}^N \tilde{g}_{i\alpha} &= {}^N \tilde{g}_{i0} = {}^N \tilde{g}_{\alpha 0} = 0. \end{aligned}$$

where $P^N = S^N$ is the sphere.

EXERCISE 9.30. *Here's another way to calculate the space-time curvatures. From* [**140**], *Lemma 4.4 we have that if we endow* $N^m \times \mathcal{I}$ *with the metric*

$$\tilde{g} = g + R dt^2,$$

then

$$\begin{aligned}
\tilde{R}^s_{pqr} &= R^s_{pqr} - R^{-1}\left(R^s_p R_{qr} - R^s_q R_{pr}\right) \\
\tilde{R}^s_{0qr} &= -\nabla_r R^s_q + \nabla^s R_{qr} - \frac{1}{2}R^{-1}\left(R_{qr}\nabla^s R - R^s_q \nabla_r R\right) \\
\tilde{R}^s_{p00} &= \frac{\partial}{\partial t}R^s_p - \frac{1}{2}\nabla_p \nabla^s R - R^m_p R^s_m + R^{-1}\left(\frac{1}{2}\frac{\partial R}{\partial t}R^s_p - \frac{1}{4}\nabla_p R \nabla^s R\right) \\
\tilde{R}^0_{0qr} &= \tilde{g}^{00}\tilde{R}_{0qr0} = \tilde{g}^{00} g_{rs}\tilde{R}^s_{q00} \\
&= R^{-1}\left(\frac{\partial}{\partial t}R^r_q - \frac{1}{2}\nabla_q \nabla_r R - R^m_q R_{mr} + R^{-1}\left(\frac{1}{2}\frac{\partial R}{\partial t}R_{qr} - \frac{1}{4}\nabla_q R \nabla_r R\right)\right)
\end{aligned}$$

Then

$$\begin{aligned}
\tilde{R}_{qr} &= R_{qr} - R^{-1}\left(R R_{qr} - R^s_q R_{sr}\right) \\
&\quad + R^{-1}\left(\frac{\partial}{\partial t}R^r_q - \frac{1}{2}\nabla_q \nabla_r R - R^m_q R_{mr}\right) \\
&\quad + R^{-2}\left(\frac{1}{2}\frac{\partial R}{\partial t}R_{qr} - \frac{1}{4}\nabla_q R \nabla_r R\right)
\end{aligned}$$

3.6. Potentially Ricci flat. Thus we should choose $b = -1$ to get ${}^N R_{ab} = O\left(N^{-1}\right)$. Note also that $R\left(ath\right) = -\frac{N}{2t}$ and when $b = -1$, we have $R\left(g\left(t\right)\right) + \frac{bN}{2t} = R\left(g\left(t\right) + ath\right)$.

Theorem 9.31 (Perelman, [**417**], §6). *If $g\left(t\right)$ and h satisfy (9.33), then the metric (whenever it is Riemannian)*

$${}^N g\left(t\right) = g\left(t\right) + \left(R\left(g\left(t\right)\right) - \frac{N}{2t}\right)dt^2 + ath$$

*is **potentially Ricci flat** in the sense that*

$${}^N R_{ab} = O\left(N^{-1}\right).$$

3.7. Letting the dimension tend to infinity. We can now relate the Levi-Civita connections and Riemann curvatures of the potentially infinite metrics with the space-time connection defined in section 1.

Corollary 9.32 (Limit of Levi-Civita connections of potentially infinite metrics). *Let*

$$\bar{\Gamma}^c_{ab} \doteqdot \lim_{N\to\infty} {}^N\Gamma^c_{ab}. \tag{9.50}$$

Then

$$\begin{aligned}\bar{\Gamma}_{ij}^{k} &= \Gamma_{ij}^{k}\\ \bar{\Gamma}_{i0}^{k} &= -R_i^k\\ \bar{\Gamma}_{00}^{k} &= -\frac{1}{2}\nabla^k R\\ \bar{\Gamma}_{00}^{0} &= -\frac{1}{2t}\\ \bar{\Gamma}_{\alpha 0}^{\gamma} &= \frac{1}{2t}\delta_\alpha^\gamma\\ \bar{\Gamma}_{\alpha\beta}^{\gamma} &= \Delta_{\alpha\beta}^{\gamma}\end{aligned}$$

and the rest of the components are 0. *In particular,*

$$\bar{\Gamma}_{ab}^{c} = \bar{\Gamma}_{ab}^{c}, \tag{9.51}$$

where $a, b, c = 0, 1, \ldots, n$ *and the* $\bar{\Gamma}$ *on the* LHS *is defined by (9.50) and the* $\bar{\Gamma}$ *on the* RHS *is defined by (9.18) and (9.19) (equality (9.51) is the reason we use the same notation.)*

Let $\bar{R}_{abc}^{d} \doteqdot \lim_{N\to\infty} {}^{N}R_{abc}^{d}$.

Corollary 9.33 (Limit of Riemann curvature tensors). *The nonzero components of* $\bar{R}_{abc}^{d}$ *are;*

$$\begin{aligned}\bar{R}_{ijk}^{\ell} &= R_{ijk}^{\ell}\\ \bar{R}_{i0k}^{\ell} &= \nabla_k R_i^\ell - \nabla^\ell R_{ik}\\ \bar{R}_{i00}^{\ell} &= \frac{\partial}{\partial t}R_i^\ell - \frac{1}{2}\nabla_i\nabla^\ell R - R_i^p R_p^\ell + \frac{1}{2t}R_i^\ell\\ \bar{R}_{\alpha\beta\gamma}^{\delta} &= R_{\alpha\beta\gamma}^{\delta}\\ \bar{R}_{\alpha 00}^{\delta} &= 0.\end{aligned}$$

In particular,

$$\bar{R}_{ibc}^{\ell} = \tilde{R}_{ibc}^{\ell} + \delta_{b0}\delta_{c0}\frac{1}{2t}R_i^\ell$$

for $i, \ell \geq 1$ *and* $b, c \geq 0$.

3.8. Potentially gradient Ricci soliton. Even when $b \neq -1$, we obtain an interesting structure for the potentially infinite space-time metrics. In particular, they are almost **potentially gradient Ricci solitons**. For the Chow-Chu potentially infinite space-time metrics, this fact was observed by Perelman in section 6 of [**417**].

Lemma 9.34 (Potentially gradient soliton). *Define* ${}^{N}f\,(t)$ *so that*

$$\boxed{\frac{\partial}{\partial t}{}^{N}f \doteqdot \frac{N}{2t}.}$$

Then for any $c \in \mathbb{R}$ we have

$$\begin{aligned}
{}^{N}R_{ij} + c\ {}^{N}\nabla_i\ {}^{N}\nabla_j\ {}^{N}f &= \left(1+\frac{1}{b}\right) R_{ij} + O\left(N^{-1}\right) - c\ {}^{N}\Gamma_{ij}^{0}\frac{\partial}{\partial t}\ {}^{N}f \\
&= \left(1+\frac{1-c}{b}\right) R_{ij} + O\left(N^{-1}\right) \\
{}^{N}R_{i0} + c\ {}^{N}\nabla_i\ {}^{N}\nabla_0\ {}^{N}f &= \left(1+\frac{1}{b}\right)\frac{1}{2}\nabla_i R + O\left(N^{-1}\right) - c\ {}^{N}\Gamma_{i0}^{0}\frac{\partial}{\partial t}\ {}^{N}f \\
&= \left(1+\frac{1-c}{b}\right)\frac{1}{2}\nabla_i R + O\left(N^{-1}\right) \\
{}^{N}R_{00} + c\ {}^{N}\nabla_0\ {}^{N}\nabla_0\ {}^{N}f &= \left(1+\frac{1}{b}\right)\frac{1}{2}\left(\frac{\partial R}{\partial t}+\frac{R}{t}\right) + O\left(N^{-1}\right) \\
&\quad + c\frac{\partial^2}{\partial t^2}\ {}^{N}f - c\ {}^{N}\Gamma_{00}^{0}\frac{\partial}{\partial t}\ {}^{N}f \\
&= \left(1+\frac{1-c}{b}\right)\frac{1}{2}\left(\frac{\partial R}{\partial t}+\frac{R}{t}\right) + O\left(N^{-1}\right) - c\frac{N}{4t^2} \\
{}^{N}R_{\alpha\beta} + c\ {}^{N}\nabla_\alpha\ {}^{N}\nabla_\beta\ {}^{N}f &= -\frac{a}{2}\left(1+\frac{1}{b}\right) h_{\alpha\beta} - c\ {}^{N}\Gamma_{\alpha\beta}^{0}\frac{\partial}{\partial t}\ {}^{N}f + O\left(N^{-1}\right) \\
&= -\frac{a}{2}\left(1+\frac{1-c}{b}\right) h_{\alpha\beta} + O\left(N^{-1}\right).
\end{aligned}$$

Recall that when $b=-1$, we chose $c=0$. More generally, we have

COROLLARY 9.35. *Given $c \in \mathbb{R} - \{1\}$, let $b = c-1 \neq 0$. Then*

$$\begin{aligned}
{}^{N}R_{ij} + c\ {}^{N}\nabla_i\ {}^{N}\nabla_j\ {}^{N}f &= O\left(N^{-1}\right) \\
{}^{N}R_{i0} + c\ {}^{N}\nabla_i\ {}^{N}\nabla_0\ {}^{N}f &= O\left(N^{-1}\right) \\
{}^{N}R_{00} + c\ {}^{N}\nabla_0\ {}^{N}\nabla_0\ {}^{N}f &= -c\frac{N}{4t^2} + O\left(N^{-1}\right).
\end{aligned}$$

If it were not for the $-c\frac{N}{4t^2}$ appearing in the last formula, the space-time metric would be potentially gradient Ricci soliton. Hence it seems that the best thing to do is to take $c=0$ and $b=-1$.

We may also let ${}^N X = \frac{\partial}{\partial t}$. (Then ${}^N\nabla_a \, {}^N X^b = {}^N\Gamma^b_{a0}$.) We have

$$\begin{aligned}
{}^N R_i^j + c \, {}^N\nabla_i \, {}^N X^j &= \left(1 - c + \frac{1}{b}\right) R_i^j + O\left(N^{-1}\right) \\
{}^N R_0^j + c \, {}^N\nabla_0 \, {}^N X^j &= \left(1 - c + \frac{1}{b}\right) \frac{1}{2} \nabla^j R + O\left(N^{-1}\right) \\
{}^N R_0^0 + c \, {}^N\nabla_0 \, {}^N X^0 &= \left(R + \frac{bN}{2t}\right)^{-1} \left(1 + \frac{1}{b}\right) \frac{1}{2} \left(\frac{\partial R}{\partial t} + \frac{R}{t}\right) \\
&\quad + c \, {}^N\Gamma^0_{00} + O\left(N^{-2}\right) \\
&= -\frac{c}{2t} + \frac{1}{N} \frac{t}{b} \left(1 + c + \frac{1}{b}\right) \left(\frac{\partial R}{\partial t} + \frac{R}{t}\right) + O\left(N^{-2}\right) \\
{}^N R_\alpha^\beta + c \, {}^N\nabla_\alpha \, {}^N X^\beta &= -\frac{1}{2t} \left(1 - c + \frac{1}{b}\right) \delta_\alpha^\beta + O\left(N^{-1}\right).
\end{aligned}$$

4. Renormalizing the space-time metric yields the ℓ-length

Now assume $\frac{d\tau}{dt} = -1$ so that the function $\tau : \mathcal{I} \to \mathcal{J} \doteqdot \tau(\mathcal{I})$ is a diffeomorphism. We now consider the space time manifold $M^n \times \mathcal{J}$ (without the extra factor P^N or S^N) with the potentially infinite metric

$$ {}^N g_{ij}(x, \tau) = g_{ij}(x, t(\tau)) \tag{9.52a} $$

$$ {}^N g_{i0}(x, \tau) = 0 \tag{9.52b} $$

$$ {}^N g_{00}(x, \tau) = R(x, t(\tau)) + \frac{N}{2\tau}. \tag{9.52c} $$

For simplicity, we shall abuse notation and write $g(\tau) \doteqdot g(t(\tau))$ and $R(x, \tau) \doteqdot R(x, t(\tau))$. Since $\frac{d\tau}{dt} = -1$ we may write

$$\frac{\partial}{\partial \tau} g_{ij}(x, \tau) = 2R_{ij}(x, \tau)$$

and forget about t. Given a path $\gamma : [\tau_1, \tau_2] \to M^n$, the length of the graph $\tilde{\gamma} : [\tau_1, \tau_2] \to M^n \times \mathcal{J}$ defined by $\tilde{\gamma}(\tau) \doteqdot (\gamma(\tau), \tau)$ is given by

$$\mathrm{L}_{({}^N g)}(\tilde{\gamma}) = \int_{\tau_1}^{\tau_2} \left|\frac{d\tilde{\gamma}}{d\tau}(\tau)\right|_{{}^N g} d\tau = \int_{\tau_1}^{\tau_2} \left(\left|\frac{d\gamma}{d\tau}\right|^2_{g(\tau)} + R + \frac{N}{2\tau}\right)^{1/2} d\tau.$$

We expand this expression in powers of N to get

$$\left(\left|\frac{d\gamma}{d\tau}\right|^2_{g(\tau)} + R + \frac{N}{2\tau}\right)^{1/2} = \left(\frac{N}{2\tau}\right)^{1/2} \left(1 + \frac{\tau}{N}\left(\left|\frac{d\gamma}{d\tau}\right|^2_{g(\tau)} + R\right) + O\left(N^{-2}\right)\right)$$

so that

$$\begin{aligned}\mathrm{L}_{\left({}^N g\right)}(\tilde{\gamma}) &= \int_{\tau_1}^{\tau_2} \left|\frac{d\tilde{\gamma}}{d\tau}(\tau)\right|_{{}^N g} d\tau \\ &= N^{1/2}\left(\sqrt{2\tau_2}-\sqrt{2\tau_1}\right) + N^{-1/2}\frac{1}{\sqrt{2}}\int_{\tau_1}^{\tau_2} \sqrt{\tau}\left(\left|\frac{d\gamma}{d\tau}\right|_{g(\tau)}^2 + R\right) d\tau \\ &\quad + O\left(N^{-3/2}\right).\end{aligned}$$

We consider highest order (in N) non-trivial term and define Perelman's **$\mathcal{L}$-length** of γ by

$$\mathcal{L}(\gamma) \doteqdot \int_{\tau_1}^{\tau_2} \sqrt{\tau}\left(\left|\frac{d\gamma}{d\tau}(\tau)\right|_{g(\tau)}^2 + R(\gamma(\tau),\tau)\right) d\tau. \tag{9.53}$$

See Chapter **??** for a discussion of the $\mathcal{L}$-length and some of its applications.

5. Space-time DeTurck's trick and fixing the measure

It is interesting to study one parameter families $g(t,s)$, $t \in \mathcal{I} \subset \mathbb{R}$ and $s \in \mathcal{J} \subset \mathbb{R}$, of solutions to the Ricci flow, where t is time and s is a real parameter (the main reference for this section is Chow-Chu [**141**]). Similar to section 8 of Chapter 2 (e.g., (2.38) and (2.39)), consider the Ricci-DeTurck flow:

$$\frac{\partial}{\partial t} g_{ij} = -2R_{ij} + \nabla_i W_j + \nabla_j W_i, \tag{9.54}$$

for $g(t,s)$ where

$$W_\ell(t,s) = g_{k\ell} g^{ij}(t,s)\left(\Gamma_{ij}^k(t,s) - \Gamma_{ij}^k(t,0)\right). \tag{9.55}$$

By (2.40), if $\left.\frac{\partial}{\partial s}\right|_{s=0} g(t,s) = v(t)$, then

$$\left.\frac{\partial}{\partial s}\right|_{s=0}\left[\left(-2\operatorname{Rc}(g) + \mathcal{L}_W g\right)(t,s)\right] = \Delta_L v(t). \tag{9.56}$$

For each s, the space-time Riemannian metrics $\hat{g}(t,s)$ on $M^n \times \mathcal{I}$ ($\hat{g}$ is positive-definite provided the constant N below is large enough) are defined by:

$$\begin{aligned}\hat{g}_{ij} &= g_{ij} \\ \hat{g}_{i0} &= W_i + \nabla_i f \\ \hat{g}_{00} &= R + |W|^2 + 2\frac{\partial f}{\partial t} + N,\end{aligned}$$

where for each $s \in \mathcal{J}$ the function $f : M^n \times \mathcal{I} \to \mathbb{R}$ is defined by

$$\boxed{f(x,t,s) \doteqdot \log\frac{d\mu(x,t,s)}{d\mu(x,t,0)}.}$$

Equivalently, the measure

$$d\mu\left(x,t,0\right)=e^{-f(x,t,s)}d\mu\left(x,t,s\right) \tag{9.57}$$

is independent of s. (In [**417**] a related idea is considered where $e^{-f}d\mu$ is independent of t. In particular, compare with (**??**) below.) The reason for the introduction of the function f is to obtain the following (see [**141**], Lemma 6.3).

LEMMA 9.36 (Variation of $\hat{g}$ is linear trace Harnack $\tilde{v}$). *If* $\frac{\partial}{\partial s}\big|_{s=0} g\left(t,s\right)=v\left(t\right)$, *then at* $s=0$

$$\boxed{\left.\frac{\partial}{\partial s}\right|_{s=0}\hat{g}\left(t,s\right)=\tilde{v}\left(t\right),}$$

where $\tilde{v}=\tilde{v}_{ab}dx^{a}\otimes dx^{b}$ *is the linear trace Harnack quadratic:*

$$\tilde{v}_{ij}\doteqdot v_{ij},\quad \tilde{v}_{k0}\doteqdot\nabla_{i}v_{ik},\quad \tilde{v}_{00}\doteqdot\nabla^{i}\nabla^{j}v_{ij}+v_{ij}R_{ij}.$$

That is, if we define the space-time vector field $\tilde{Y}$ by $\tilde{Y}^{i}=Y^{i}$ and $\tilde{Y}^{0}=1$, then

$$\tilde{v}_{ab}\tilde{Y}^{a}\tilde{Y}^{b}=\bar{Z}\left[v\right]\left(Y\right)$$

which appears in (8.68).

Now let's compute the evolution equation for $v\left(t\right)$

$$\begin{aligned}\frac{\partial}{\partial t}v_{ij}&=\frac{\partial}{\partial t}\left(\left.\frac{\partial}{\partial s}\right|_{s=0}g_{ij}\right)=\left.\frac{\partial}{\partial s}\right|_{s=0}\left(\frac{\partial}{\partial t}g_{ij}\right)\\&=\left.\frac{\partial}{\partial s}\right|_{s=0}\left(-2R_{ij}+\nabla_{i}W_{j}+\nabla_{j}W_{i}\right)\\&=\Delta_{L}v_{ij},\end{aligned}$$

so that $v\left(t\right)$ satisfies the Lichnerowicz Laplacian heat equation with respect to the solution $g\left(t,0\right)$. This is the reason for considering the set of equations (9.54)-(9.55).

We also can use the above lemma to compute the evolution equation for the linear trace Harnack quadratic $\tilde{v}$. Let $\tilde{R}_{ab}$ and $\tilde{\Gamma}_{ab}^{c}$ denote the space-time Ricci tensor and connection as defined in section 1. One can derive a space-time version of the variation formula (9.56) provided we define the space-time 1-form $\tilde{W}$ by

$$\tilde{W}_{i}\doteqdot W_{i}\quad\text{and}\quad\tilde{W}_{0}\doteqdot\sum_{p=1}^{n}\nabla^{p}W_{p}.$$

(Note the similarity of the above formula with (9.30) and (9.31).) With this definition, one can show that (see Theorem 5.1 of [**141**]):

LEMMA 9.37 (Variation of space-time DeTurck-Ricci tensor is $\tilde{\Delta}_{L}\tilde{v}$).

$$\left.\frac{\partial}{\partial s}\right|_{s=0}\left(-2\tilde{R}_{ab}+\tilde{\nabla}_{a}\tilde{W}_{b}+\tilde{\nabla}_{b}\tilde{W}_{a}\right)=\tilde{\Delta}_{L}\tilde{v}_{ab},$$

where

$$\tilde{\Delta}_L \tilde{v}_{ab} \doteqdot \Delta v_{ab} + 2\tilde{g}^{ce}\tilde{g}^{df}\tilde{R}_{cabd}\tilde{v}_{ef} - \tilde{g}^{cd}\tilde{R}_{ac}\tilde{v}_{bd} - \tilde{g}^{cd}\tilde{R}_{bc}\tilde{v}_{ad}.$$

One can also show that the space-time Riemannian metrics $\hat{g}$ solve some sort of space-time Ricci-DeTurck flow to first order in the sense of the following (see Lemma 6.4 of [**141**]).

LEMMA 9.38 ($\frac{\partial}{\partial t}\hat{g}$ is space-time DeTurck-Ricci tensor mod $o(s)$).

$$\left(\frac{\partial}{\partial t}\hat{g}_{ab}\right)(s,t) = \left(-2\tilde{R}_{ab} + \tilde{\nabla}_a \tilde{W}_b + \tilde{\nabla}_b \tilde{W}_a\right)(s,t) + o(s),$$

where $\left.\frac{\partial}{\partial s}\right|_{s=0} o(s) = 0$.

Combining the above results we obtain:

THEOREM 9.39 (Chow and S.-C. Chu 1996). *The linear trace Harnack quadratic satisfies the space-time Lichnerowicz Laplacian heat equation:*

$$\boxed{\frac{\partial}{\partial t}\tilde{v}_{ab} = \tilde{\Delta}_L \tilde{v}_{ab}.}$$

PROOF. We compute

$$\begin{aligned}\frac{\partial}{\partial t}\tilde{v}_{ab} &= \frac{\partial}{\partial t}\left(\left.\frac{\partial}{\partial s}\right|_{s=0}\hat{g}_{ab}\right) = \left.\frac{\partial}{\partial s}\right|_{s=0}\left(\frac{\partial}{\partial t}\hat{g}_{ab}\right) \\ &= \left.\frac{\partial}{\partial s}\right|_{s=0}\left(\left(-2\tilde{R}_{ab} + \tilde{\nabla}_a \tilde{W}_b + \tilde{\nabla}_b \tilde{W}_a\right) + o(s)\right) \\ &= \tilde{\Delta}_L \tilde{v}_{ab}.\end{aligned}$$

□

This gives another (space-time) derivation of the linear trace Harnack calculation.

6. Notes and commentary

§3. In §6.1 of [**417**] $g_{\alpha\beta}$ is the metric on S^N with constant sectional curvature $\frac{1}{2N}$. In our presentation, as a special case we have $g_{\alpha\beta}$ is the metric on S^N with constant sectional curvature $\frac{1}{2(N-1)}$ so that its Ricci curvatures are equal to $\frac{1}{2}$ and scalar curvature equal to $\frac{N}{2}$. The difference in the curvatures is $O\left(N^{-1}\right)$.

Let $\tilde{g} \doteqdot {}^N g(t)$ on $M^n \times \mathcal{I} \times P^N$ be the metric defined in (9.34) with $b = -1$ and let f be a function on $M^n \times \mathcal{I} \times P^N$ independent of the P^N variables. Define the map $\phi : \phi^{-1}\left(M^n \times \mathcal{I} \times P^N\right) \to M^n \times \mathcal{I} \times P^N$ by

$$\phi(x,t,y) \doteqdot \left(x, \left(1 - \frac{2f}{N}\right)t, y\right).$$

Let ${}^m\tilde{g} \doteqdot \phi^*\tilde{g}$. Using

$$\frac{\partial\phi^i}{\partial x^j} = \delta^i_j, \quad \frac{\partial\phi^\alpha}{\partial y^\beta} = \delta^\alpha_\beta,$$
$$\frac{\partial\phi^0}{\partial t} = 1 - \frac{2}{N}\left(f + t\frac{\partial f}{\partial t}\right), \quad \frac{\partial\phi^0}{\partial x^i} = -\frac{2t}{N}\frac{\partial f}{\partial x^i},$$

we compute

$$\begin{aligned}
{}^m\tilde{g}_{ij} &= \tilde{g}_{ij} + \frac{\partial\phi^0}{\partial x^i}\frac{\partial\phi^0}{\partial x^j}\tilde{g}_{00} \\
&= \tilde{g}_{ij} + O\left(N^{-1}\right) \\
{}^m\tilde{g}_{i0} &= \frac{\partial\phi^0}{\partial x^i}\frac{\partial\phi^0}{\partial t}\tilde{g}_{00} = \frac{\partial f}{\partial x^i}\left(1 - \frac{2}{N}\left(f + t\frac{\partial f}{\partial t}\right)\right)\left(\frac{1}{1-\frac{2f}{N}} - \frac{2tR}{N}\right) \\
&= \frac{\partial f}{\partial x^i} + O\left(N^{-1}\right) \\
{}^m\tilde{g}_{00} &= \left(\frac{\partial\phi^0}{\partial t}\right)^2\tilde{g}_{00} = \left(1 - \frac{2}{N}\left(f + t\frac{\partial f}{\partial t}\right)\right)^2\left(R - \frac{N}{2t\left(1-\frac{2f}{N}\right)}\right) \\
&= \tilde{g}_{00} + 2\left(\frac{\partial f}{\partial t} + \frac{f}{t}\right) + O\left(N^{-1}\right) \\
&= -\frac{N}{2t} + R + 2\frac{\partial f}{\partial t} + \frac{f}{t} + O\left(N^{-1}\right) \\
{}^m\tilde{g}_{\alpha\beta} &= \tilde{g}_{\alpha\beta}
\end{aligned}$$

and the rest of the components are zero. In the above formulas, the LHS is evaluated at (x,t,y) whereas the RHS is evaluated at $\left(x, \left(1-\frac{2f}{N}\right)t, y\right)$. So

$$\tilde{g}_{00} = R - \frac{N}{2t\left(1-\frac{2f}{N}\right)} = R - \frac{N}{2t} - \frac{f}{t} + O\left(N^{-1}\right)$$

and also, for example, the last formula reads more explicitly as

$${}^m\tilde{g}_{\alpha\beta}(x,t,y) = \tilde{g}_{\alpha\beta}\left(x, \left(1-\frac{2f}{N}\right)t, y\right) = \left(1-\frac{2f}{N}\right)\tilde{g}_{\alpha\beta}(x,t,y).$$

Note also that assuming a bound on $\frac{\partial R}{\partial t}$ we have

$$R\left(x, \left(1-\frac{2f}{N}\right)t\right) = R(x,t) + O\left(N^{-1}\right).$$

Taking into account all of our (including sign) conventions, the above calculations agree with §6.2 of [**417**].

EXERCISE 9.40. *Assuming* $\frac{\partial R}{\partial t}$ *is bounded, verify that*

$${}^m\tilde{g}_{00}(x,t,y) = \tilde{g}_{00}(x,t,y) + 2\frac{\partial f}{\partial t} + \frac{f}{t} + O\left(N^{-1}\right).$$

Now consider time-dependent diffeomorphisms $\psi_t : M^n \to M^n$ with

$$\frac{\partial}{\partial t}\psi_t = -\left(\psi_t\right)_* \left(\operatorname{grad}_{^m g}\left(f \circ \psi_t\right)\right) = -\operatorname{grad}_{^m \tilde{g}} f$$

(the second inequality is (1.24)) so that

$$\left(\psi_t^{-1}\right)_* \frac{\partial}{\partial t}\psi_t = -\operatorname{grad}_{^m g}\left(f \circ \psi_t\right)$$

and the metrics ${}^m g_{ij}(t) \doteqdot \psi_t^* \left({}^m \tilde{g}_{ij}(t)\right)$ satisfy the modified Ricci flow (see Remark 1.22):

$$\begin{aligned}\frac{\partial}{\partial t}\left({}^m g_{ij}\right) &= \psi_t^* \left(\frac{\partial}{\partial t}\ {}^m \tilde{g}_{ij}(t)\right) + \mathcal{L}_{\left(\psi_t^{-1}\right)_* \frac{\partial}{\partial t}\psi_t}\ {}^m g_{ij} \\ &= -2\left({}^m R_{ij} + {}^m \nabla_i\ {}^m \nabla_j \left(f \circ \psi_t\right)\right).\end{aligned}$$

Define $\Psi : (\phi \circ \Psi)^{-1}\left(M^n \times \mathcal{I} \times P^N\right) \to \phi^{-1}\left(M^n \times \mathcal{I} \times P^N\right)$ by

$$\Psi(x,t,y) = \left(\psi_t(x), t, y\right).$$

We consider the metric ${}^m g \doteqdot \Psi^* \left({}^m \tilde{g}\right)$ on $(\phi \circ \Psi)^{-1}\left(M^n \times \mathcal{I} \times P^N\right)$. Above we have seen the pure space components ${}^m g_{ij}$. Note

$$\begin{aligned}&\frac{\partial \Psi^i}{\partial x^j} = \frac{\partial \psi_t^i}{\partial x^j}, \quad \frac{\partial \Psi^i}{\partial t} = -\left(\operatorname{grad}_{^m \tilde{g}} f\right)^i, \\ &\frac{\partial \Psi^0}{\partial t} = 1, \quad \frac{\partial \Psi^\alpha}{\partial y^\beta} = \delta_\beta^\alpha.\end{aligned}$$

We now calculate the rest of the components of ${}^m g$. First we have

$$\begin{aligned}{}^m g_{i0} &= \frac{\partial \Psi^j}{\partial x^i}\left(\frac{\partial \Psi^0}{\partial t}\ {}^m \tilde{g}_{j0} + \frac{\partial \Psi^k}{\partial t}\ {}^m \tilde{g}_{jk}\right) \\ &= \frac{\partial \psi_t^j}{\partial x^i}\left(\frac{\partial f}{\partial x^j} + O\left(N^{-1}\right) - \left(\operatorname{grad}_{^m \tilde{g}} f\right)^k\ {}^m \tilde{g}_{jk}\right) \\ &= O\left(N^{-1}\right).\end{aligned}$$

Hence, with respect to ${}^m g$, $\frac{\partial}{\partial t}$ is orthogonal (mod N^{-1}) to the space slices $M^n \times \{t\} \times P^N$. Next we have

$$\begin{aligned}{}^m g_{00} &= {}^m \tilde{g}_{00} + 2\frac{\partial \Psi^i}{\partial t}\frac{\partial \Psi^0}{\partial t}\ {}^m \tilde{g}_{i0} + \frac{\partial \Psi^i}{\partial t}\frac{\partial \Psi^j}{\partial t}\ {}^m \tilde{g}_{ij} \\ &= {}^m \tilde{g}_{00} - \left|\operatorname{grad}_{^m \tilde{g}} f\right|_{^m \tilde{g}}^2 \\ &= R - \frac{N}{2t\left(1 - \frac{2f}{N}\right)} + 2\left(\frac{\partial f}{\partial t} + \frac{f}{t}\right) - \left|\operatorname{grad}_{^m \tilde{g}} f\right|_{^m \tilde{g}}^2 + O\left(N^{-1}\right) \\ &= -\frac{N}{2t} - \left(R + 2\Delta f - |\nabla f|^2 + \frac{f-n}{-t}\right) + O\left(N^{-1}\right),\end{aligned}$$

where the last equality assumes $\frac{\partial f}{\partial t} = -\Delta f - R + |\nabla f|^2 - \frac{n}{2t}$. In the last line, the term in brackets is $-t$ times Perelman's Harnack quantity

$$-t\left(R + 2\Delta f - |\nabla f|^2\right) + f - n = V_{-1}$$

(see (**??**) and (**??**).)

§**4**. The following observations on the relation between Huisken's monotonicity formula for the mean curvature flow and Perelman's renormalization procedure applied to Chow and Chu's space-time metric for evolving hypersurfaces are due to S.-C. Chu. Given a 1-parameter family of metrics $g(t)$, $t \in \mathcal{I}$, on a manifold M^n and functions $\beta(t) : M^n \to \mathbb{R}$, we define the metric g_β on $\tilde{M}^{n+1} \doteqdot M^n \times \mathcal{I}$ by ([**142**])

$$g_\beta(x,t) \doteqdot g(x,t) + \beta^2(x,t)\, dt^2.$$

We consider the family of hypersurfaces given by the time slices $M_t \doteqdot M^n \times \{t\} \subset \tilde{M}^{n+1}$. A choice of unit normal vector field to M_t is

$$\nu \doteqdot -\frac{1}{\beta}\frac{\partial}{\partial t}.$$

The hypersurfaces M_t parametrized by the maps $X_t : M^n \to \tilde{M}^{n+1}$ defined by $X_t(x) \doteqdot (x,t)$ are evolving by the flow

$$\frac{\partial}{\partial t}X_t = -\beta\nu.$$

This implies the metrics are evolving by (compare (1.84))

$$\frac{\partial}{\partial t}g_{ij} = -2\beta h_{ij},$$

where h_{ij} is the second fundamental form of $M_t \subset \tilde{M}^{n+1}$. One way of seeing this formula is from

$$\frac{1}{\beta}h_{ij} = (\Gamma_\beta)^0_{ij} = -\frac{1}{2}(g_\beta)^{00}\frac{\partial}{\partial x^0}(g_\beta)_{ij} = -\frac{1}{2\beta^2}\frac{\partial}{\partial t}g_{ij}$$

where $x^0 = t$. Hence

$$\beta h_{ij} = R_{ij}. \tag{9.58}$$

Consider the special case where $\beta(t)^2 = R(t)$ is the scalar curvature of $g(t)$. Tracing (9.58) we get $\beta H = R$ so that $\beta = H$ and the hypersurfaces M_t satisfy the mean curvature flow: $\frac{\partial}{\partial t}X_t = -H\nu$.

Now we consider the more general setting of hypersurfaces evolving in a Riemannian manifold. Given $\left(P^{n+1}, g\right)$, let $X_t : M^n \to P^{n+1}$, $t \in \mathcal{I}$, parametrize a 1-parameter family of hypersurfaces $M_t = X_t(M^n)$ evolving in their normal directions

$$\frac{\partial}{\partial t}X_t = -\beta\nu,$$

where $\beta(t) : M^n \to \mathbb{R}$ are arbitrary functions. We consider the product metric $g + Ndt^2$ on $P^{n+1} \times \mathcal{I}$. The **space-time track** is defined by

$$\tilde{M}^{n+1} \doteqdot \{(x,t) : x \in M_t,\ t \in \mathcal{I}\} \subset P^{n+1} \times \mathcal{I}.$$

We parametrize this by the map

$$\tilde{X} : M^n \times \mathcal{I} \to P^{n+1} \times \mathcal{I}$$

defined by

$$\tilde{X}(p,t) \doteqdot (X_t(p), t).$$

Let ${}^N g$ denote the induced metric on $\tilde{M}^{n+1}$. Its components

$${}^N g_{ab} \doteqdot \left\langle \frac{\partial \tilde{X}}{\partial x^a}, \frac{\partial \tilde{X}}{\partial x^b} \right\rangle_{g+Ndt^2} = \left\langle \frac{\partial X_t}{\partial x^a}, \frac{\partial X_t}{\partial x^b} \right\rangle_g + N\delta_{a0}\delta_{b0},$$

where $a, b \geq 0$ are given by

$$\begin{aligned} {}^N g_{ij} &= g_{ij} \\ {}^N g_{i0} &= 0 \\ {}^N g_{00} &= \beta^2 + N, \end{aligned}$$

where $i, j \geq 1$. The second fundamental form ${}^N h_{ij}$ of the space-time track $\tilde{M}^{n+1}$ with respect to the ambient metric $g + Ndt^2$ on $P^{n+1} \times \mathcal{I}$ is given by

$${}^N h_{ab} = \left(1 + \frac{\beta^2}{N}\right)^{-1/2} \begin{cases} h_{ab} & \text{if } a, b \geq 1 \\ \frac{\partial \beta}{\partial x^b} & \text{if } a = 0,\ b \geq 1 \\ \frac{\partial \beta}{\partial t} & \text{if } a = b = 0 \end{cases}.$$

Given a vector $V \in T_p M^n$, we define the vector $\tilde{V} \in T_{(p,t)}(M^n \times \mathcal{I})$ by

$$\begin{aligned} \tilde{V}^i &= V^i \\ \tilde{V}^0 &= 1. \end{aligned}$$

Then

$${}^N h_{ab} \tilde{V}^a \tilde{V}^b = \frac{\partial \beta}{\partial t} + 2\frac{\partial \beta}{\partial x^i} V^i + h_{ij} V^i V^j.$$

This is **Andrews' Harnack quadratic** for curvature flows of hypersurfaces (see [**13**]); when $\beta = H$ is the mean curvature we have Hamilton's quadratic [**269**].

Now, following Perelman, we renormalize length function associated to the metric (similar to what we did in section 4) on $M^n \times \mathcal{J}$ (we switch from $\mathcal{I}$ to $\mathcal{J}$ when we consider the time parameter to be τ instead of t)

$${}^N g(x,\tau) \doteqdot g(x,\tau) + \left(\beta^2(x,\tau) + \frac{N}{2\tau}\right) d\tau^2,$$

where $\frac{d\tau}{dt} = -1$ and $g(\tau) = g(t(\tau))$ is the pulled back metric on M^n by X_τ of the induced metric on $M_\tau \doteqdot X_\tau(M^n) \subset P^{n+1}$. We may also think of this metric as defined on an open subset of P^{n+1} by pushing forward by the diffeomorphism $(x, \tau) \mapsto X_\tau(x)$. Let $\gamma : [0, \tau_0] \to M^n$ be a path and define the path $\bar{\gamma} : [0, \tau_0] \to P^{n+1}$ by

$$\bar{\gamma}(\tau) \doteqdot X_\tau(\gamma(\tau)) \in M_\tau$$

so that $(\gamma(\tau), \tau) \in M^n \times \mathcal{J}$ corresponds to the point $\bar{\gamma}(\tau) \in M_\tau \subset P^{n+1}$. We have

$$\mathrm{L}_{({}^N g)}(\bar{\gamma}) = \int_0^{\tau_0} \left(\left| \frac{d\gamma}{d\tau} \right|^2_{g(\tau)} + \beta^2 + \frac{N}{2\tau} \right)^{1/2} d\tau.$$

Again, motivated by carrying out the expansion of $\mathrm{L}_{({}^N g)}(\bar{\gamma})$ in powers of N, and considering highest order non-trivial term, we define the **$\mathcal{L}$-length** of γ by

$$\begin{aligned} \mathcal{L}(\gamma) &\doteqdot \int_0^{\tau_0} \sqrt{\tau} \left(\left| \frac{d\gamma}{d\tau}(\tau) \right|^2_{g(\tau)} + \beta^2(\gamma(\tau), \tau) \right) d\tau \\ &= \int_0^{\tau_0} \sqrt{\tau} \left| \frac{d\bar{\gamma}}{d\tau}(\tau) \right|^2_g d\tau. \end{aligned}$$

(The equality holds since $\iota^* g = g_\beta$, where $\iota : M^n \times \mathcal{J} \to P^{n+1}$ is defined by $\iota(x, \tau) \doteqdot X_\tau(x)$.) Making the change of variables $\sigma = 2\sqrt{\tau}$, we have

$$\mathcal{L}(\gamma) = \int_0^{2\sqrt{\tau_0}} \left| \frac{d\bar{\gamma}}{d\tau}(\sigma) \right|^2_g d\sigma.$$

This is the energy of the path $\bar{\gamma}(\sigma)$ and assuming that τ_0, $\gamma(0) = p$ and $\gamma(\tau_0) = q$ are fixed, $\mathcal{L}(\gamma)$ is minimized by a constant speed geodesic and

$$L(q, \tau_0) \doteqdot \inf_\gamma \mathcal{L}(\gamma) = \frac{d_g(p, q)^2}{2\sqrt{\tau_0}}.$$

Let $\ell(q, \tau_0) \doteqdot \frac{1}{2\sqrt{\tau_0}} L(q, \tau_0)$. Then

$$\ell(q, \tau_0) = \frac{d_g(p, q)^2}{4\tau_0}.$$

Hence Huisken's monotonicity (**??**) is the analogue of the monotonicity of the **reduced volume** (see Volume 2.) That is, if $P^{n+1} = \mathbb{R}^{n+1}$, then the monotone quantity in (**??**) is

$$\int_{X_t} (4\pi\tau)^{-n/2} e^{-\frac{|x|^2}{4\tau}} d\mu = \int_{M^n} (4\pi\tau)^{-n/2} e^{-\ell} d\mu,$$

where $\ell = \frac{1}{2\sqrt{\tau}} \inf_\gamma \int_0^\tau \sqrt{\bar{\tau}} \left(\left| \frac{d\gamma}{d\bar{\tau}} \right|^2 + \beta^2 \right) d\bar{\tau}$.

§5. In Friedan [**213**], p. 402, when M^n is a homogeneous space the **zero loop term** $-\varepsilon g_{ij}$ in the expansion of $\beta(g)$ is the gradient of the functional

$$\Phi_0(g) \doteqdot -2\varepsilon \log \left(\frac{d\mu_g}{dm} \right),$$

where dm is any G-invariant volume form on M^n. The **one loop term** R_{ij} is the gradient of the functional

$$\Phi_1(g) \doteqdot -R.$$

The two loop term is the gradient of

$$\Phi_2(g) \doteqdot -\frac{1}{4}|\mathrm{Rm}|^2 .$$

Index

Aleksandrov reflection method, 213
ancient limit, 230
ancient solution, 151
  3-dimensional has nonnegative sectional curvature, 190
  Type I, 238
  Type II, 238
ancient solutions
  have nonnegative scalar curvature, 230
  on surfaces, 240
Andrews' Harnack quadratic, 315
Andrews' Poincare inequality, 210
arithmetic-geometric mean inequality, 94
asymptotic cone, 245
asymptotic volume ratio, 48

Bernstein-Bando-Shi estimates, 179
Bianchi identity
  first, 9
  second, 9
Bianchi operator, 125
Bieberbach theorem, 196
bisectional curvature
  nonnegative, 275
Bishop volume comparison theorem, 32
blow up rate
  of curvature, 231
Bryant soliton, 163
bumpy metrics, 178
Busemann function, 34

Calabi's trick, 262
Cartan structure equations, 22, 156
Cartan-Hadamard theorem, 1
changing distances
  bounds on , 165
Christoffel symbols, 4, 112
  evolution of, 113, 287
  variation formula, 112
cigar soliton, 155, 240
classical Harnack, 260
coframe field, 21
Cohn-Vossen inequality, 242, 245
commutation formula, 14
cone angle, 162
Conformal dilation, 216
conformal Killing vector field, 207
conjugate point, 30
connection
  Levi-Civita, 3
connection 1-forms, 21
constant sectional curvature, 9
contracted second Bianchi identity, 9
convex set, 135
coordinates
  geodesic , 37
cosmological constant, 292
covariant derivative, 3
  acting on tensors, 8
cross curvature tensor, 226
cross-curvature flow, 226
curvature
  Gauss, 22
curvature 2-forms, 21
curvature blow-up rate
  lower bound for, 232
curvature gap estimate
  for ancient solutions, 237
  of finite time singularity, 231
  of immortal solution, 232
curvature operator, 132
  nonnegative, 194
  positive, 132
curvature tensor
  Riemann, 5
curve shortening flow, 231
cut locus, 30
cut point, 30

degenerate metric, 288

degenerate neck pinch, 237
deRham cohomology group, 123
deRham local splitting theorem, 196
deRham Theorem, 123
derivative of curvature estimates, 179
derivatives of curvature estimate, 178
determinant of the Laplacian, 82
DeTurck's trick, 119
  space-time version, 309
differential Harnack estimate, 259, 266
dilating about a singularity, 229
distance function
  laplacian of, 46
distance sphere, 40, 43
distributions
  sense of, 31
divergence, 25
divergence theorem, 27
doubling time estimate, 177

eigenfunction, 70
eigenvalue, 70
Einstein convention, 4
Einstein tensor, 9
Einstein-Hilbert functional, 112
eternal solution, 151
euclidean space
  characterization of, 32
evolution equation
  Ricci tensor, 128
  Riemann curvature tensor, 130
  scalar curvature, 104
  volume form, 111, 129
existence
  long time, 178
expanding Ricci soliton, 160
exponential map, 29, 37
  Jacobian of, 39
exterior covariant derivative, 23, 95

first Bianchi identity, 9
first fundamental form, 98
first variation of arc length formula, 51
frame adapted to a hypersurface, 23
frame bundle
  orthonormal, 275
Frechet derivative, 213
fundamental solution of the heat equation, 62

gamma function, 80
Gauss curvature, 22
Gauss equations, 24
Gauss Lemma, 39
geodesic, 5
  constant speed, 5, 55
geodesic coordinates, 37
geodesic line, 34
geodesic ray, 34
Geometrization Conjecture, 2
geometry at spatial infinity, 245
gradient flow
  Einstein-Hilbert functional, 112
gradient of Hamilton's entropy, 213
gradient Ricci soliton, 151, 265
Green's function, 61
Gromoll-Meyer Theorem, 60

Hölder's inequality, 94
Hamilton's entropy
  gradient of, 213
Hamilton, R., 2
Hamilton-Ivey estimate, 187
harmonic form, 124
harmonic map, 120
Harnack estimate
  differential, 259
  integral, 267
  interpolating, 268
  matrix, 268
  trace, 266
heat ball, 284
heat equation
  one-dimensional, 239
  weak subsolution, 107
heat kernel, 62, 79
heat operator, 78
higher derivative of curvature estimates, 178
Hodge Decomposition Theorem, 124
Hodge Laplacian, 123
Hodge laplacian, 116
Hodge star operator, 123
homogeneous space, 169
  locally, 160
Huisken's monotonicity formula
  is analogous to the reduced volume, 316

immortal solution, 151, 160
injectivity radius, 31
integral Harnack estimate, 267
interpolating Harnack estimate, 268
invariant under parallel translation, 135
isometry, 12
isoperimetric constant, 221, 225
isoperimetric ratio, 221

Jacobi equation, 29, 55
Jacobi field, 29, 55
Jacobian of the exponential map, 39
Jensen's inequality, 94

Kähler-Ricci soliton, 162
Killing vector field, 12
  conformal, 207
Klingenberg's injectivity radius estimate, 241, 251

L69
L^2-metric, 212
L^2 Sobolev inequality, 95
Lambert-W function, 162
Laplace-Beltrami operator, 25
laplacian, 25
  Hodge, 116
  Lichnerowicz, 114
  rough, 26
laplacian comparison theorem, 31
laplacian of the distance function, 46
largest curvature scale, 230
length of a geodesic
  evolution of, 220
Levi-Civita connection, 3
Li-Yau estimate, 259
Lichnerowicz laplacian, 114
Lie algebra, 132
Lie algebra square, 132
Lie bracket, 3
Lie derivative, 10
Lie group
  unimodular, 169
line
  geodesic, 34
linear connection, 95
linear trace estimate, 268
linear trace Harnack estimate, 278
Liouville type theorem, 261
local derivative of curvature estimates, 179
locally conformally flat, 213
locally homogeneous, 160
logarithmic Sobolev inequality, 95
long time existence, 178
L^p-norm, 94

manifold
  Riemannian, 2
matrix Harnack estimate, 268, 271
Matrix,
  The, 1
maximum principle
  for systems, 135
  for tensors, 127
  weak, 105
mean curvature, 24, 40
Mean Value Inequality, 35
measurable function, 70
Mellin transform, 81
metric
  degenerate, 288
  Riemannian, 2
metric on the cotangent bundle, 6
modified Ricci flow, 161
moving frame, 21
moving frames, 156
Myers' theorem, 1

nonnegative bisectional curvature, 275
nonnegative curvature operator, 194
normal coordinates, 113
normalized Ricci flow, 129
normalized Yamabe flow, 214
notation, xxv
null eigenvector assumption, 127

one loop term, 317
open problem, 103, 174, 199, 212, 213, 231, 233, 238, 245, 255, 256, 258
orthonormal frame, 21
orthonormal frame bundle, 275

parallel
  vector field along a path, 5
parallel curves, 221
Perelman, G., xi
Pig,
  yellow, 95
Poincaré Conjecture, 2
Poincaré duality, 125
Poincare inequality
  Andrews', 210
point picking, 239
polar coordinate system, 39
positive curvature operator
  conjecture, 198
potentially infinite
  dimensions, 295
  space-time metric, 295
potentially Ricci flat
  space-time metric, 305
preconverges, 182, 233
preservation
  of nonnegative curvature operator, 136
  of nonnegative Ricci curvature, 136
  of nonnegative scalar curvature, 105

of Ricci pinching, 137
preservation
of lower bound of scalar curvature, 136
of nonnegative Ricci curvature, 129
product log function, 162

rapidly forming singularity, 231
Rauch Comparison Theorem, 50
ray
geodesic, 34
Rayleigh principle, 71
reaction-diffusion equation, 129
reduced volume, 316
renormalizing the space-time metric, 308
Riccatti equation, 41
Ricci calculus, 8
Ricci curvature, 6
Ricci flow
equation, 103
for degenerate metrics, 288
game of , 111
modified, 161
normalized, 129
Ricci flow with cosmological term, 292
Ricci flow with surgeries, 101
Ricci identities, 14
Ricci pinching
is preserved, 137
Ricci pinching
improves, 138
Ricci soliton
expanding, 160
gradient, 151, 265
triviality of, 209
Ricci tensor, 6
evolution equation, 128
variation formula, 114
Ricci tensor
nonnegativity is preserved, 128
Ricci–De Turck flow, 119
Riemann curvature operator, 132
Riemann curvature tensor, 5
components, 114
evolution equation, 130
Riemann zeta function, 80
Riemannian manifold, 2
Riemannian measure, 30
Riemannian metric, 2
Rosenau solution, 158
rotationally symmetric metric, 42
rough laplacian, 26

scalar curvature, 6
evolution equation, 104
variation formula, 104
second Bianchi identity, 9
contracted, 9
second fundamental form, 23, 40
second variation of arc length formula, 52
sectional curvature, 6
constant, 9
self-similar, 259
sense of distributions, 31
Sharafutdinov retraction, 37
short time existence, 103
singularity
rapidly forming, 231
slowly forming, 231
Type I, 231
Type IIa, 231
Type IIb, 231
Type III, 231
singularity formation, 229
singularity model, 230
slowly forming singularity, 231
smallest space scale, 230
soliton
cigar, 155
space form, 42
space of metrics, 212
space-time, 288
is potentially gradient soliton, 306
potential infinite dimensions, 295
potential infinite metric, 295
Ricci soliton equation, 293
Ricci soliton identities, 293
space-time connection, 289
space-time curvature, 290
is the matrix Harnack, 294
space-time geometry, 287
space-time laplacian
tends to heat operator, 298
space-time metric
is potentially Ricci flat, 305
renormalization of, 308
space-time track, 315
spatial infinity
geometry at, 245
sphere
distance, 43
spherical space form, 2
stereographic projection, 216
strong maximum principle, 72
for systems, 195
structure equations

Cartan, 22, 156
summation convention
Einstein, 4
support functions
in the sense of, 54
surface entropy, 210

tensors
as functions on the orthonormal frame bundle, 275
three-manifolds with positive Ricci curvature, 2, 130
Thurston, W., 2
total scalar curvature functional, 112
totally umbillic, 43
trace Harnack estimate, 266
trace of the heat operator, 79
Type I singularity, 231, 233
Type IIa singularity, 231, 234, 238
Type IIb singularity, 231, 235
Type III singularity, 231, 236

Uhlenbeck's trick, 132, 275
Uniform equivalence of the metrics
sufficient condition, 180
unimodular Lie group, 169

variation formula
Christoffel symbols, 112
Einstein-Hilbert functional, 112
Ricci tensor, 114
scalar curvature, 104
vector field
Killing, 12
volume form, 38
evolution equation, 129
local coordinate formula, 111

warped product metric, 163
weak maximum principle, 105
for tensors, 127
weak subsolution
of the heat equation, 107
Weingarten map, 24
Weyl asymptotic formula, 70
Weyl's Estimate, 99
Witten's black hole, 156

Yamabe flow, 213, 214
Yamabe invariant, 215
Yamabe soliton, 209

zero loop term, 316
zeta function, 80, 82

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